NTNU

TMA4215 Numerical Mathematics — Fall 2016

PROBLEM SET 8

1 a) Implement the power method given in eq. (5.17) in the textbook by Quarteroni *et al.* Test it on the matrix

$$A = \begin{bmatrix} -2 & -2 & 3 \\ -10 & -1 & 6 \\ 10 & -2 & -9 \end{bmatrix}$$

and use $\boldsymbol{x}^{(0)} = (1, 0, 0)$ as a starting value. Compare your result with that of MATLAB's built-in function `eig`.

b) When you are sure to have a working code, try it on the matrix

$$B = \begin{bmatrix} 5 & 1 & -1 \\ 1 & 11 & 7 \\ -1 & 7 & 11 \end{bmatrix}$$

using the same $\boldsymbol{x}^{(0)}$ as above. What is the result after 20 iterations? After 100? Explain what you observe.

c) Now implement the shifted inverse method—also called the inverse power method—in eq. (5.28) in Quarteroni *et al.* and test your function on the matrices from a) and b).

d) Apply the QR algorithm in eq. (5.32) in Quarteroni *et al.* on the two matrices from a) and b) and explain your findings.

2 Let I_n be the $n \times n$ identity matrix, $\boldsymbol{v} \in \mathbb{R}^n$ and $\theta \in \mathbb{R}$. Prove that the following matrices are orthogonal:

a) $\Omega = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

b) $Q = I_n - 2\frac{1}{\boldsymbol{v}^\top \boldsymbol{v}} \boldsymbol{v}\boldsymbol{v}^\top$.

Also, show that Q is symmetric.

3 Find by hand the reduced QR factorization of the matrix

$$A = \begin{bmatrix} 4 & 4 \\ 0 & 2 \\ 3 & 3 \end{bmatrix}$$

using the Gram–Schmidt orthogonalization process.

4 The Householder QR factorization of $A \in \mathbb{R}^{n\times m}$, where $n \geq m$, is given by the following algorithm:

for $k = 1$ to m **do**

$\quad \boldsymbol{x} \leftarrow A_{k:n,k}$

$\quad \boldsymbol{v}_k := \boldsymbol{x} + \operatorname{sign}(x_1)\|x\|_2 \boldsymbol{e}_1$

$\quad \boldsymbol{v}_k \leftarrow \boldsymbol{v}_k / \|\boldsymbol{v}_k\|_2$

$\quad A_{k:n,k:m} \leftarrow A_{k:n,k:m} - 2\boldsymbol{v}_k \left(\boldsymbol{v}_k^\top A_{k:n,k:m}\right)$

end for

Here $A_{i:j,k:\ell}$—omitting colon(s) if $i = j$ and/or $k = \ell$—denotes the elements of A in accordance with MATLAB's indexing convention. This algorithm transforms A into the upper trapezoidal matrix $R \in \mathbb{R}^{n\times m}$ in the QR decomposition, and is implemented in the attached file `householder.m`. It does not, however, produce Q, but we know that

$$Q = Q_1 Q_2 \cdots Q_m,$$

where

$$Q_1 = I_n - 2\boldsymbol{v}_1\boldsymbol{v}_1^\top \qquad \text{and} \qquad Q_k = \begin{bmatrix} I_{k-1} & 0 \\ 0 & I_{n-k+1} - 2\boldsymbol{v}_k\boldsymbol{v}_k^\top \end{bmatrix} \quad \text{for} \quad k = 2, \ldots, m.$$

a) Apply the algorithm first by hand and then with `householder.m` to the matrix in Exercise 3. Also, find Q_1 and Q_2, and compute Q. Compare with the results from MATLAB's built-in function `qr`.

b) Let $\boldsymbol{x} \in \mathbb{R}^n$ and remember that $Q \in \mathbb{R}^{n\times n}$. Show that the product $Q\boldsymbol{x}$ is performed by the following procedure:

for $k = m$ downto 1 **do**

$\quad \boldsymbol{x}_{k:n} \leftarrow \boldsymbol{x}_{k:n} - 2\boldsymbol{v}_k(\boldsymbol{v}_k^\top \boldsymbol{x}_{k:n})$

end for

How can we use this to form Q itself?

c) Extend `householder.m` to also return Q, and compare your result with `qr`.

d) Let A be the Hilbert matrix of dimension 8 – you can build it in MATLAB by typing `A = hilb(8)`. Find the QR decomposition of A both via the Gram–Schmidt orthogonalization process—included in the attached file `GramSchmidt.m` for convenience—and the Householder transformations. How well will $Q^\top Q$ approximate I_8 in the two cases?