NTNU

TMA4215 Numerical Mathematics — Fall 2016

Problem Set 9

1 A sequence $\{\boldsymbol{x}^{(k)}\}$ from an iterative method is said to converge to $\boldsymbol{x}$ with rate/order $p > 1$ in a norm $\|\cdot\|$ if

$$\frac{\|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}\|}{\|\boldsymbol{x}^{(k)} - \boldsymbol{x}\|^p} \to \mu \geq 0 \qquad \text{as} \qquad k \to \infty.$$

We write x_k instead of $\boldsymbol{x}^{(k)}$ when x is a scalar. Also, when $p = 1$, we say that $\{\boldsymbol{x}^{(k)}\}$ converges to $\boldsymbol{x}$

- linearly if $\mu \in (0, 1)$;
- superlinearly, that is, asymptotically faster than linear, if $\mu = 0$;
- or sublinearly, that is, asymptotically slower than linear, if $\mu = 1$ but still $\boldsymbol{x}^{(k)} \to \boldsymbol{x}$ [1].

Suggestively, convergence is called quadratic when $p = 2$, cubic when $p = 3$, and so on. Note, however, that convergence rates can be nonintegral—for example, the classical secant method for root-finding is locally convergent with order equal to the golden ratio.

What is the best (largest) order of convergence for the following sequences?

a) $x_k = 3^{-k}$. b) $x_k = (-1)^k (k+1)^{-2}$. c) $x_k = \pi^{-5^k}$.

2 Recall that Newton's method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

applied to a twice continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ is locally quadratically convergent provided that the root α is simple. If, however, α has multiplicity $m > 1$, in other words, if

$$f(\alpha) = f'(\alpha) = \cdots = f^{(m-1)}(\alpha) = 0 \qquad \text{and} \qquad f^{(m)}(\alpha) \neq 0$$

for a sufficiently smooth $f \in C^m(\mathbb{R})$, then the rate of convergence is just linear. In this exercise, you will establish that the modified Newton scheme

$$x_{k+1} = x_k - m\frac{f(x_k)}{f'(x_k)} \tag{$*$}$$

retains the quadratic convergence speed in the presence of a root α with multiplicity m.

a) Show that α is a fixed point of the function $\phi(x) = x - mf(x)/f'(x)$. *Hint:* l'Hôpital's rule.

b) Prove that $\phi'(\alpha) = 0$. *Hint:* Write f as $f(x) = (x-\alpha)^m h(x)$ for some function h satisfying $h(\alpha) \neq 0$ before you calculate $f'(x)$ and $f''(x)$.

[1] This follows implicitly for all the other cases by the ratio test for sequences.

c) Let $e_k = x_k - \alpha$ be the error at step k and show that

$$e_{k+1} = \tfrac{1}{2}\phi''(\xi)e_k^2$$

for some ξ between x_k and α. Now deduce that $(*)$ achieves quadratic convergence speed.

3 Let $\boldsymbol{G}\colon \mathbb{R}^3 \to \mathbb{R}^3$ be given as

$$\boldsymbol{G}(\boldsymbol{x}) = \big(g_1(\boldsymbol{x}), g_2(\boldsymbol{x}), g_3(\boldsymbol{x})\big) = \begin{bmatrix} \frac{1}{6}\big(1+2\cos(x_1x_2)\big) \\ \frac{1}{9}\sqrt{x_1^2+\sin x_3+1.06}-0.1 \\ \frac{1}{20}\left(1-\mathrm{e}^{-x_1x_2}\right)-\frac{\pi}{6} \end{bmatrix},$$

where $\boldsymbol{x} = (x_1, x_2, x_3)$. Show that the fixed-point iteration $\boldsymbol{x}^{(k+1)} = \boldsymbol{G}\big(\boldsymbol{x}^{(k)}\big)$ converges towards a unique fixed point for all starting vectors $\boldsymbol{x}^{(0)}$ in the closed cube

$$D = \big\{\boldsymbol{x} \in \mathbb{R}^3 : -1 \le x_1, x_2, x_3 \le 1\big\}.$$

Also, verify the result numerically.

4 Consider the system of equations

$$x_1^2 + x_2^2 = 1;$$
$$x_1^3 - x_2 = 0,$$

which has two solutions—one in the region $-1 \le x_1, x_2 \le 0$ and the other one in $0 \le x_1, x_2 \le 1$. Try a fixed-point scheme based on the formulation

$$x_1 = \sqrt[3]{x_2};$$
$$x_2 = \sqrt{1-x_1^2},$$

and show—with explanation—that it converges for suitable starting values. How would you select the starting values?

Hint: It is simpler to do the analysis if you consider two subsequent iterations as one. That is, consider instead the scheme $\boldsymbol{x}^{(k+2)} = \boldsymbol{G}\big(\boldsymbol{G}\big(\boldsymbol{x}^{(k)}\big)\big)$ with the appropriate $\boldsymbol{G}$. This decouples the iterations into two scalar cases.