NTNU

TMA4215 Numerical Mathematics — Fall 2016

Problem Set 9

1 A sequence $\{\boldsymbol{x}^{(k)}\}$ from an iterative method is said to converge to $\boldsymbol{x}$ with rate/order $p > 1$ in a norm $\|\cdot\|$ if

$$\frac{\|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}\|}{\|\boldsymbol{x}^{(k)} - \boldsymbol{x}\|^p} \to \mu \geq 0 \qquad \text{as} \qquad k \to \infty.$$

We write x_k instead of $\boldsymbol{x}^{(k)}$ when x is a scalar. Also, when $p = 1$, we say that $\{\boldsymbol{x}^{(k)}\}$ converges to $\boldsymbol{x}$

- linearly if $\mu \in (0, 1)$;
- superlinearly, that is, asymptotically faster than linear, if $\mu = 0$;
- or sublinearly, that is, asymptotically slower than linear, if $\mu = 1$ but still $\boldsymbol{x}^{(k)} \to \boldsymbol{x}$ [1].

Suggestively, convergence is called quadratic when $p = 2$, cubic when $p = 3$, and so on. Note, however, that convergence rates can be nonintegral—for example, the classical secant method for root-finding is locally convergent with order equal to the golden ratio.

What is the best (largest) order of convergence for the following sequences?

a) $x_k = 3^{-k}$. b) $x_k = (-1)^k (k+1)^{-2}$. c) $x_k = \pi^{-5^k}$.

Solution. a) Since

$$\frac{|x_{k+1}|}{|x_k|^p} = 3^{k(p-1)-1} \xrightarrow[k\to\infty]{} \begin{cases} \infty & \text{if } p > 1; \\ 1/3 & \text{if } p = 1, \end{cases}$$

it follows that $\{x_k\}$ converges linearly to 0 with $\mu = 1/3$.

b) We have $x_k \to 0$ and

$$\frac{|x_{k+1}|}{|x_k|^p} = \left(\frac{(k+1)^p}{k+2}\right)^2 \xrightarrow[k\to\infty]{} \begin{cases} \infty & \text{if } p > 1; \\ 1 & \text{if } p = 1, \end{cases}$$

so the rate of convergence is sublinear.

c) This time

$$\frac{|x_{k+1}|}{|x_k|^p} = \pi^{5^k(p-5)},$$

which yields that the order of convergence is quintic, with $\mu = 1$.

[1] This follows implicitly for all the other cases by the ratio test for sequences.

2 Recall that Newton's method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

applied to a twice continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ is locally quadratically convergent provided that the root α is simple. If, however, α has multiplicity $m > 1$, in other words, if

$$f(\alpha) = f'(\alpha) = \cdots = f^{(m-1)}(\alpha) = 0 \qquad \text{and} \qquad f^{(m)}(\alpha) \neq 0$$

for a sufficiently smooth $f \in C^m(\mathbb{R})$, then the rate of convergence is just linear. In this exercise, you will establish that the modified Newton scheme

$$x_{k+1} = x_k - m\frac{f(x_k)}{f'(x_k)} \tag{$*$}$$

retains the quadratic convergence speed in the presence of a root α with multiplicity m.

a) Show that α is a fixed point of the function $\phi(x) = x - mf(x)/f'(x)$. *Hint:* l'Hôpital's rule.

Solution. By l'Hôpital's rule and the multiplicity of α,

$$\frac{f(\alpha)}{f'(\alpha)} = \frac{f'(\alpha)}{f''(\alpha)} = \cdots = \frac{f^{(m-1)}(\alpha)}{f^{(m)}(\alpha)} = 0,$$

because $f^{(m-1)}(\alpha) = 0$ and $f^{(m)}(\alpha) \neq 0$. Thus $\phi(\alpha) = \alpha$.

b) Prove that $\phi'(\alpha) = 0$. *Hint:* Write f as $f(x) = (x-\alpha)^m h(x)$ for some function h satisfying $h(\alpha) \neq 0$ before you calculate $f'(x)$ and $f''(x)$.

Solution. Routine calculations give

$$\phi'(x) = 1 - m + m\frac{f(x)f''(x)}{f'(x)^2},$$

with

$$f'(x) = (x-\alpha)^{m-1}\left[mh(x) + (x-\alpha)h'(x)\right]$$

and

$$f''(x) = (x-\alpha)^{m-2}\left[m(m-1)h(x) + 2m(x-\alpha)h'(x) + (x-\alpha)^2 h''(x)\right].$$

Since $h(\alpha) \neq 0$, we find that

$$\begin{aligned}\frac{f(x)f''(x)}{f'(x)^2} &= \frac{h(x)\left[m(m-1)h(x) + 2m(x-\alpha)h'(x) + (x-\alpha)^2 h''(x)\right]}{\left[mh(x) + (x-\alpha)h'(x)\right]^2} \\ &\xrightarrow[x\to\alpha]{} \frac{m(m-1)}{m^2} = 1 - \frac{1}{m},\end{aligned}$$

and so $\phi'(\alpha) = 0$, as desired.

c) Let $e_k = x_k - \alpha$ be the error at step k and show that

$$e_{k+1} = \tfrac{1}{2}\phi''(\xi)e_k^2$$

for some ξ between x_k and α. Now deduce that $(*)$ achieves quadratic convergence speed.

Solution. Notice first that $e_{k+1} = x_{k+1} - \alpha = \phi(x_k) - \phi(\alpha)$ from $(*)$, the definition of ϕ and a). Hence, by Taylor expansion and b),

$$e_{k+1} = \phi'(\alpha)(x_k - \alpha) + \tfrac{1}{2}\phi''(\xi)(x_k - \alpha)^2 = \tfrac{1}{2}\phi''(\xi)e_k^2$$

for some ξ between x_k and α. Consequently, quadratic convergence rate follows from

$$\lim_{k\to\infty} \frac{e_{k+1}}{e_k^2} = \tfrac{1}{2}\phi''(\alpha),$$

where we have used that $\xi \to \alpha$ as $x_k \to \alpha$ and the continuity of ϕ''.

$\boxed{3}$ Let $\boldsymbol{G}\colon \mathbb{R}^3 \to \mathbb{R}^3$ be given as

$$\boldsymbol{G}(\boldsymbol{x}) = \big(g_1(\boldsymbol{x}), g_2(\boldsymbol{x}), g_3(\boldsymbol{x})\big) = \begin{bmatrix} \frac{1}{6}\big(1 + 2\cos(x_1x_2)\big) \\ \frac{1}{9}\sqrt{x_1^2 + \sin x_3 + 1.06} - 0.1 \\ \frac{1}{20}\big(1 - \mathrm{e}^{-x_1x_2}\big) - \frac{\pi}{6} \end{bmatrix},$$

where $\boldsymbol{x} = (x_1, x_2, x_3)$. Show that the fixed-point iteration $\boldsymbol{x}^{(k+1)} = \boldsymbol{G}\big(\boldsymbol{x}^{(k)}\big)$ converges towards a unique fixed point for all starting vectors $\boldsymbol{x}^{(0)}$ in the closed cube

$$D = \big\{\boldsymbol{x} \in \mathbb{R}^3 : -1 \le x_1, x_2, x_3 \le 1\big\}.$$

Also, verify the result numerically.

Solution. If we can show that $\boldsymbol{G}$ is a contraction—that is, a Lipschitz function with Lipschitz constant less then 1—satisfying $\boldsymbol{G}(D) \subseteq D$, then Banach's fixed-point theorem guarantees the existence of a unique fixed point in D.

It can be seen that

$$\begin{aligned}
g_1(1,1,x_3) &\approx 0.34 &&< g_1(x_1,x_2,x_3) \le 0.5 &&= g_1(0,x_2,x_3);\\
g_2(0,x_2,-1) &\approx -0.048 &&< g_2(x_1,x_2,x_3) < 0.09 &&\approx g_2(1,x_2,1);\\
g_3(-1,1,x_3) &\approx -0.61 &&< g_3(x_1,x_2,x_3) < -0.49 &&\approx g_2(1,1,x_3),
\end{aligned}$$

so that, indeed, $\boldsymbol{G}(D) \subseteq D$. Furthermore, from the mean value theorem we know that contractivity of $\boldsymbol{G}$ is assured provided

$$\max_{i=1,2,3} \sum_{j=1}^{3} \left|\frac{\partial g_i}{\partial x_j}(\boldsymbol{x})\right| < 1 \quad \text{for all} \quad \boldsymbol{x} \in D.$$

Some calculations yield that

$$\left|\frac{\partial g_1}{\partial x_1}\right| < 0.281, \qquad \left|\frac{\partial g_1}{\partial x_2}\right| < 0.281, \qquad \left|\frac{\partial g_1}{\partial x_3}\right| = 0,$$
$$\left|\frac{\partial g_2}{\partial x_1}\right| < 0.067, \qquad \left|\frac{\partial g_2}{\partial x_2}\right| = 0, \qquad \left|\frac{\partial g_2}{\partial x_3}\right| < 0.119,$$
$$\left|\frac{\partial g_3}{\partial x_1}\right| < 0.136, \qquad \left|\frac{\partial g_3}{\partial x_2}\right| < 0.136, \qquad \left|\frac{\partial g_3}{\partial x_3}\right| = 0,$$

for all $\boldsymbol{x} \in D$, which means that

$$\max_{i=1,2,3} \sum_{j=1}^{3} \left|\frac{\partial g_i}{\partial x_j}(\boldsymbol{x})\right| < \max\{0.562, 0.186, 0.272\} = 0.562 < 1.$$

Hence, $\boldsymbol{G}$ has a unique fixed point in D.

The fixed point equals $\boldsymbol{x} = \frac{1}{6}(3, 0, -\pi)$ and the following script in MATLAB demonstrates that the fixed-point iteration converges for any $\boldsymbol{x}_0 \in D$.

```
G = @(x) [(1 + 2 * cos(x(1) .* x(2))) / 6; ...
          sqrt(x(1).^2 + sin(x(3)) + 1.06) / 9 - .1; ...
          (1 - exp(-x(1) .* x(2))) / 20 - (pi / 6)];
fixedpt = [.5; 0; -pi / 6];

% Run fixed-point iteration with random starting points
% and compute maximum absolute error of all tests.
tests = 100;
testpoints = -1 + 2 * rand(3, tests);
for n = 1:tests
    for k = 1:13
        testpoints(:, n) = G(testpoints(:, n));
    end
end
max_abs_error = max(max(abs(testpoints - repmat(fixedpt, 1, tests))))
```

Noticeably, convergence is obtained after just 13 iterations for all the tests.

4 Consider the system of equations

$$x_1^2 + x_2^2 = 1;$$
$$x_1^3 - x_2 = 0,$$

which has two solutions—one in the region $-1 \le x_1, x_2 \le 0$ and the other one in $0 \le x_1, x_2 \le 1$. Try a fixed-point scheme based on the formulation

$$x_1 = \sqrt[3]{x_2};$$
$$x_2 = \sqrt{1 - x_1^2},$$

and show—with explanation—that it converges for suitable starting values. How would you select the starting values?

Hint: It is simpler to do the analysis if you consider two subsequent iterations as one. That is, consider instead the scheme $\boldsymbol{x}^{(k+2)} = \boldsymbol{G}\big(\boldsymbol{G}\big(\boldsymbol{x}^{(k)}\big)\big)$ with the appropriate $\boldsymbol{G}$. This decouples the iterations into two scalar cases.

Solution. The fixed-point iteration for two steps is given by

$$x_1^{(k+1)} = \sqrt[3]{x_2^{(k)}}, \qquad x_1^{(k+2)} = \sqrt[6]{1-\big(x_1^{(k)}\big)^2},$$
$$x_2^{(k+1)} = \sqrt{1-\big(x_1^{(k)}\big)^2}, \qquad x_2^{(k+2)} = \sqrt{1-\big(x_2^{(k)}\big)^{2/3}}.$$

Since $x_1^{(k+2)}$ and $x_2^{(k+2)}$ only depend on $x_1^{(k)}$ and $x_2^{(k)}$, respectively, the original iteration can be viewed as fixed-point iterations on the two separate scalar equations

$$x = g_1(x) = \sqrt[6]{1-x^2} \qquad \text{and} \qquad x = g_2(x) = \sqrt{1-x^{2/3}}.$$

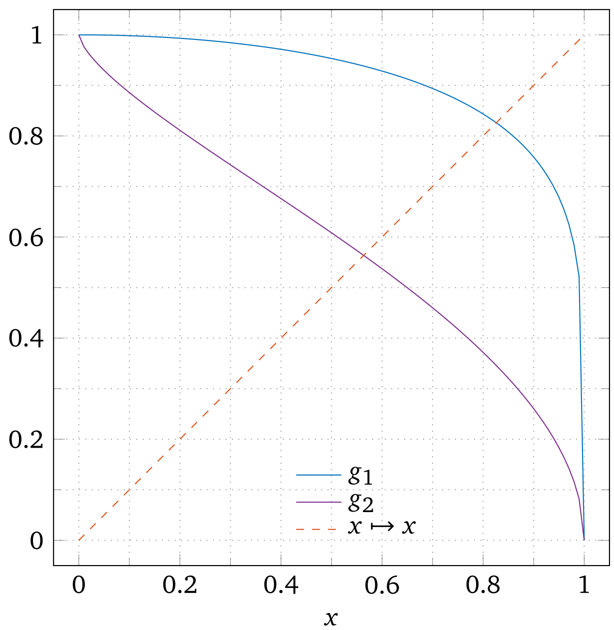

Figure 1: Graphs of g_1 and g_2 and their fixed-points.

We now intend to use Banach's fixed point theorem on the g_i's, and for each of these we need to find an interval $[a,b]$ such that $g_i([a,b]) \subseteq [a,b]$ and $|g_i'(x)| < 1$ for all $x \in [a,b]$.

Practically, we start by locating the fixed points graphically. Figure 1 shows that g_1 has a fixed point near 0.8, and g_2 one near 0.5. Considering g_1 first, we calculate

$$g_1'(x) = -\frac{x}{3(1-x^2)^{5/6}},$$

which implies that $|g_1'(x)| < 1$ at least when $0 \le x \le 0.87$. This interval, however, does not satisfy $g_1([0,0.87]) \subseteq [0,0.87]$. But since g_1 is monotonically decreasing, some trial and error reveals that

$$g_1([0.76, 0.87]) \subseteq [0.76, 0.87].$$

Similarly, we can show that the two conditions are satisfied for g_2 on the interval $[0.22, 0.80]$.

In conclusion, the system of equations has a unique fixed point in the region

$$D = \left\{\boldsymbol{x} \in \mathbb{R}^2 : 0.76 \le x_1 \le 0.87, 0.22 \le x_2 \le 0.80\right\},$$

and the iterations converge for all starting values in D.