TMA4215 Numerical Mathematics — Fall 2016

NTNU

Problem Set 10

1 Determine the values of (a,b,c) such that

$$S(x) = \begin{cases} S_0(x) = x^3 & x \in [0,1) \\ S_1(x) = \frac{1}{2}(x-1)^3 + a(x-1)^2 + b(x-1) + c & x \in [1,3] \end{cases}$$

is a cubic spline. Is it a natural cubic spline?

Solution. Obviously, both S_0 and S_1 are polynomials of the degree 3. We then have to check that $S \in C^2[0,3]$, that is

$$\begin{aligned} S_0(1) &= S_1(1) && \Rightarrow && c = 1 \\ S_0'(1) &= S_1'(1) && \Rightarrow && b = 3 \\ S_0''(1) &= S_1''(1) && \Rightarrow && a = 3 \end{aligned}$$

This is a *natural* cubic spline if in addition $S''(0) = S''(3) = 0$. The first condition is satisfied, the second not, so this not a natural cubic spline.

2 Given the data set:

x	1.2	1.5	1.6	2.0	2.2
$f(x)$	0.4275	1.139	0.8736	−0.9751	−0.1536

Find the linear spline $L(x)$ interpolating the data set, and give the value for $L(1.8)$.

Solution. The linear spline $L(x)$ consist of piecewise linear polynomials, and it is continuous. Thus

$$L_i(x) = f(t_i) + (f(t_{i+1}) - f(t_i))/(t_{i+1} - t_i)(x - t_i)$$

such that

$$\begin{aligned} L_0(x) &= 2.3717x - 2.4185, && 1.2 \le x \le 1.5 \\ L_1(x) &= -2.6540x + 5.1200, && 1.5 \le x \le 1.6 \\ L_2(x) &= -4.6218x + 8.6284, && 1.6 \le x \le 2.0 \\ L_3(x) &= 4.1075x - 9.1901, && 2.0 \le x \le 2.2 \end{aligned}$$

and in particular $L_2(1.8) = -0.05075$.

3

a) Write down the 4 cubic Bernstein polynomials, $b_{3,i}(t)$, $i = 0,1,2,3$.

Solution.

$$b_{3,0}(t) = (1-t)^3, \qquad b_{3,1}(t) = 3t(1-t)^2, \qquad b_{3,2}(t) = 3t^2(1-t), \qquad b_{3,3}(t) = t^3.$$

b) Given the control points

$$\mathbf{P}_0 = (0,0), \qquad \mathbf{P}_1 = (1,2), \qquad \mathbf{P}_2 = (2,-1), \qquad \mathbf{P}_3 = (1,0).$$

Write up and plot the corresponding Bezier curve,

$$\mathbf{B}(t) = \sum_{i=0}^{3} \mathbf{P}_i b_{3,i}(t).$$

Prove that the straight line between the first two control points is tangential to the curve in $\mathbf{P}_0$, similar is the straight line between the last two point tangential to the curve in $\mathbf{P}_3$.

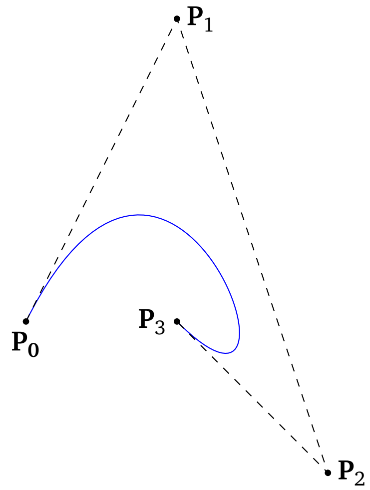

Figure 1: The Bezier curve $\mathbf{B}(t)$ (blue) and the control points.

Solution.

$$\mathbf{B}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 3t(1-t)^2 + 6t^2(1-t) + t^3 \\ 6t(1-t)^2 - 3t^2(1-t) \end{pmatrix} = \begin{pmatrix} -2t^3 + 3t \\ 9t^3 - 15t^2 + 6t \end{pmatrix}$$

see figure 1. For the last part: The slope of the curve between $\mathbf{P}_0$ and $\mathbf{P}_1$ is 2, and the slope between $\mathbf{P}_3$ and $\mathbf{P}_2$ is -1. If $y'(t) \neq 0$ then $dy/dx = y'(t)/x'(t)$. We have

$$\frac{dB}{dt} = \begin{pmatrix} -6t + 3 \\ 27t^2 - 30t + 6 \end{pmatrix}$$

So, at $t = 0$ we have $dy/dx = 2$ and at $t = 1$ we have $dy/dx = -1$.

c) Assume that you know the value of a function $f(x)$ and its derivative $f'(x)$ for two values of x, say x_0 and x_1, with $h = x_1 - x_0 > 0$. Let $x = x_0 + th$ and let

$$B(x) = B(x_0 + th) = \sum_{i=0}^{3} a_i b_{3,i}(t).$$

Find a_i, $i = 0,1,2,3$ such that

$$B(x_i) = f(x_i), \qquad B'(x_i) = f'(x_i), \qquad i = 1,2$$

where $B' = dB/dx$.

Solution. We have that

$$B'(x) = \frac{1}{h}\frac{dB}{dt} = \frac{1}{h}\sum_{i=0}^{3} a_i \frac{db_{3,i}}{dt}$$

where

$$\frac{db_{3,0}}{dt} = -3(1-t)^2, \qquad \frac{db_{3,1}}{dt} = 3(1-t)(1-3t), \qquad \frac{db_{3,2}}{dt} = 3t(2-3t), \qquad \frac{db_{3,1}}{dt} = 3t^2.$$

The following conditions has to be fullfilled:

$$\begin{aligned} B(x_0) &= f(x_0) = a_0, & B(x_1) &= f(x_1) = a_3, \\ B'(x_0) &= f'(x_0) = 3h(a_1 - a_0), & B'(x_1) &= f'(x_1) = 3h(a_3 - a_2), \end{aligned}$$

giving

$$a_0 = f(x_0), \qquad a_1 = f(x_0) + \frac{1}{3}hf'(x_0), \qquad a_2 = f(x_1) - \frac{1}{3}hf'(x_1), \qquad a_3 = f(x_1).$$