TMA4220 Numerical solution of partial differential equations by element methods

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Exercise 1. Consider the problem $-u_{xx} = 1$, $0 < x < 1$, $u(0) = u(1) = 0$.

(a) Derive the exact solution $u = \frac{1}{2}x(1-x)$.

(b) Show by explicit calculation that $\delta J_v(u) = \int_0^1 (u_x v_x - v)dx = 0$ for all (smooth) v such that $v(0) = v(1) = 0$.

(c) Compute $J(u)$.

(d) Consider the functions $w_n(x) = a_n\, sin(n\pi x)$, $n = 1, 2, 3, ...$ These functions are infinitely differentiable on $x \in [0, 1]$, and they all satisfy $w_n(0) = w_n(1) = 0$. Show that $J(w_n) = \frac{a_n^2}{4}(n\pi)^2 - \frac{2a_n}{n\pi}$ when $n = 1, 3, 5, ...$, i.e., when n is odd.

(e) Consider the case $n = 1$. Find the value of the amplitude a_1 which minimizes $J(w_1)$. How does this value of the amplitude of a_1 compare with the maximum (or "amplitude") of the exact solution u?

(f) With the value of the amplitude a_1 computed in (e), show that $J(w_1) > J(u)$. Is there a big difference between these minima?

(g) If we define $\hat{w}$ as $\hat{w}(x) = a_1\, sin(\pi x) + a_3\, sin(3\pi x)$, is the statement statement $J(u) < \min_{a_1,a_3} J(\hat{w}) < \min_{a_1} J(w_1)$ true or false? Explain your answer (hint: you should not have to do any explicit computation of $J(\hat{w})$).