TMA4220 Numerical solution of partial differential equations by element methods
Problem set 8

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Exercise 1. We consider the one-dimensional, steady convection-diffusion problem

$$
\begin{aligned}
-\kappa\, u_{xx} + U\, u_x &= f \qquad in \quad \Omega = (0,1)\ , &(1)\\
u(0) &= 0\ , &(2)\\
u(1) &= 1\ . &(3)
\end{aligned}
$$

Assume that we solve this problem using linear finite elements. The element matrix $\underline{A}_h^k$ associated with element k can be expressed as

$$
\underline{A}_h^k = \frac{\kappa}{h^k}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{U}{2}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\ . \tag{4}
$$

Here, h^k is the length of element T_h^k, $k = 1, ..., K$. Derive this element matrix. Hint: complete the discussion in the notes dated March 9, 2004.

Exercise 2. Consider the model problem discussed in class:

$$
\begin{aligned}
-\kappa\, u_{xx} - U\, u_x &= 0 \qquad in \quad \Omega = (0,\infty)\ , &(5)\\
u(0) &= 1\ , &(6)\\
u(\infty) &= 0\ . &(7)
\end{aligned}
$$

Assume that we discretize both the diffusion term and the convection term using second-order central differences. For what grid Peclet numbers do we get oscillations in the sense that the numerical solution will become both positive and negative? Hint: just inspect the exact form of the numerical solution.

Exercise 3. Again, consider the model problem in Exercise 2. Show that first-order upwinding is equivalent to center differences applied to an equation in which the parameter $\varepsilon = \kappa/U$ is replaced by $\varepsilon + \Delta x/2$. What is the maximum grid Peclet number using first-order upwinding?