TMA4220
Numerical solution of partial differential equations
by element methods

Suggested solutions to Problem Set 1[1]

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Exercise 1

(Separate page made by Einar Rønquist)

a)

$$
\begin{aligned}
-u_{xx} &= 1, \qquad u(0) = u(1) = 0 \\
-u_x &= x + A \\
-u &= \frac{1}{2}x^2 + Ax + B
\end{aligned}
$$

$$
\begin{aligned}
u(0) = 0 \Rightarrow B &= 0 \\
u(1) = 0 \Rightarrow A &= -\frac{1}{2}
\end{aligned}
$$

$$
u = -\frac{1}{2}x^2 + \frac{1}{2}x = \underline{\frac{1}{2}x(1-x)}
$$

b)

$$
\begin{aligned}
\delta J_v(u) &= \int_0^1 (u_x v_x - v)\,\mathrm{d}x \\
&= [\underset{0}{\cancel{v}}\, u_x]_0^1 + \int_0^1 (-v u_{xx} - v)\,\mathrm{d}x \\
&= -\int_0^1 v \underbrace{(u_{xx} + 1)}_{=0} \mathrm{d}x = \underline{0}
\end{aligned}
$$

c)

$$
\begin{aligned}
J(u) &= \frac{1}{2}\int_0^1 u_x^2\,\mathrm{d}x - \int_0^1 u\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^1 (\frac{1}{2} - x)^2\,\mathrm{d}x - \int_0^1 \frac{1}{2}x(1-x)\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^1 (\frac{1}{4} - x + x^2)\,\mathrm{d}x - \frac{1}{2}\int_0^1 (x - x^2)\,\mathrm{d}x \\
&= \frac{1}{2}[\frac{1}{4}x - \frac{1}{2}x^2 + \frac{1}{3}x^3]_0^1 - \frac{1}{2}[\frac{1}{2}x^2 - \frac{1}{3}x^3]_0^1 \\
&= \underline{-\frac{1}{24}}
\end{aligned}
$$

d)

$$w_n = a_n \sin n\pi x, \qquad w_x = a_n n\pi \cos n\pi x$$

$$\begin{aligned} J(w_n) &= \frac{1}{2}a_n^2 \int_0^1 n^2\pi^2 \cos^2 n\pi x - a_n \int_0^1 \sin n\pi x \, \mathrm{d}x \\ &= \frac{1}{2}(n\pi)a_n^2 \int_0^{n\pi} \cos^2 \varphi \, \mathrm{d}\varphi - \frac{1}{n\pi}a_n \int_0^{n\pi} \sin\varphi \, \mathrm{d}\varphi \end{aligned}$$

Recall that: $\cos^2\varphi = \frac{1+\cos 2\varphi}{2} \Rightarrow \int_0^{n\pi} \cos^2\varphi \, \mathrm{d}\varphi = \frac{1}{2}n\pi$

$$\int_0^{n\pi} \sin\varphi \, \mathrm{d}\varphi = [-\cos\varphi]_0^{n\pi} = -\cos n\pi + \cos 0 = 1 - (-1)^n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$$

$$\Downarrow$$

$$\begin{aligned} J(w_n) &= \frac{1}{2}(n\pi)a_n^2 \frac{n\pi}{2} - \frac{a_n}{n\pi}2, \qquad n \text{ odd} \\ &= \underline{\frac{1}{4}a_n^2(n\pi)^2 - \frac{2a_n}{n\pi}}, \qquad n \text{ odd} \end{aligned}$$

e)

$\underline{n=1}$: $\qquad J(w_1) = \frac{1}{4}a_1^2\pi^2 - \frac{2a_1}{\pi}$

$$\frac{\mathrm{d}J}{\mathrm{d}a_1} = \frac{2}{4}a_1\pi^2 - \frac{2}{\pi} = 0 \Rightarrow a_1 = \frac{4}{\pi^3} = 0.129\ldots$$

$$\begin{aligned} w_1 &= \frac{4}{\pi^3}\sin\pi x \Rightarrow w_1(\frac{1}{2}) = \frac{4}{\pi^3} \simeq \underline{0.129} \\ u &= \frac{1}{2}x(1-x) \Rightarrow u(\frac{1}{2}) = \frac{1}{8} = \underline{0.125} \end{aligned}$$

f)

$$\begin{aligned} J(w_1) &= \frac{1}{4}(\frac{4}{\pi^3})^2\pi^2 - \frac{2}{\pi}\frac{4}{\pi^3} = \frac{4}{\pi^4} - \frac{8}{\pi^4} = -\frac{4}{\pi^4} \\ &= -0.04106\ldots \\ &\simeq -\frac{1}{24.35} > J(u) = -\frac{1}{24} \end{aligned}$$

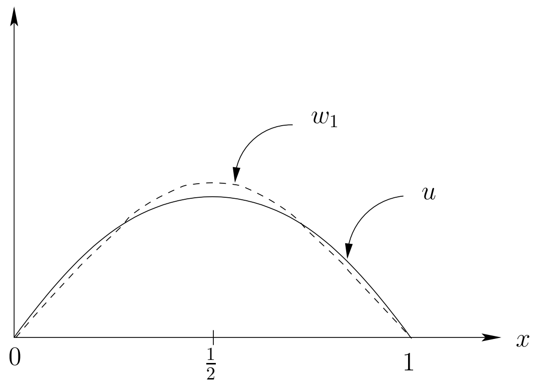
g)

True.

Our basis has now been expanded from $\{\sin\pi x\}$ to $\{\sin\pi x, \sin 3\pi x\}$.

We therefore expect

$$\min_{a_1,a_3} J(\hat{w}) \leq \min_{a_1} J(w_1)$$



However, we also expect

$$J(u) < \min_{a_1, a_3} J(\hat{w})$$

since only the limit

$$\lim_{N\to\infty} \min_{\{a_i\}_{n=1}^N} J\left(\sum_{n=1}^{N} a_n \sin(n\pi x)\right) = J(u)$$

Exercise 3

(March 19 & 31, 2003)
a)

$J : Y \to \mathbb{R}, \quad J(w) = \frac{1}{2}a(w,w) - l(w)$

$$\begin{aligned} J(u+v) &= \frac{1}{2}a(u+v, u+v) - l(u+v) \\ &= \frac{1}{2}a(u,u) - l(u) + \frac{1}{2}a(u,v) + \frac{1}{2}a(v,u) - l(v) + \frac{1}{2}a(v,v) \\ &= J(u) + \frac{1}{2}(a(u,v) + a(v,u)) - l(v) + \frac{1}{2}a(v,v) \end{aligned}$$

Since $a(u,v) = a(v,u)$:

$$J(u+v) = J(u) + \underbrace{(a(u,v) - l(v))}_{\delta J_v(u)} + \frac{1}{2}a(v,v)$$

If $\delta J_v(u) = 0 \quad \forall v \in Y$

$$J(u+v) = J(u) + \frac{1}{2}a(v,v) > J(u) \qquad \begin{matrix} \forall v \in Y \\ v \neq 0 \end{matrix}$$

Hence, the minimizer, u, must satisfy

$$\underline{a(u,v) = l(v)} \qquad \forall v \in Y$$

b)

$Y = \mathbb{R}^n, \qquad J(w) = \frac{1}{2}w^T G w - w^T F$

Identify

$$a(w,v) = w^T G v$$
$$l(w) = w^T F$$

Note that $a(w,v)$ is an SPD bilinear form over $Y = \mathbb{R}^n$ because G is an SPD matrix $G \in \mathbb{R}^{nxn}$. For example,

$$a(w,v) = w^T G v = (w^T G v)^T = v^T G^T w = v^T G w = a(v,w)$$
$$a(w,w) = w^T G w > 0 \qquad \forall w \in \mathbb{R}^n, w \neq 0$$

From the result in **a)**, we immediately find that the minimizer $u \in Y$ of $J(w)$ satisfies

$$a(u,v) = l(v) \qquad \forall v \in Y$$

that is,

$$u^T G v = v^T F \qquad \forall v \in \mathbb{R}^n$$
$$v^T G u = v^T F \qquad \forall v \in \mathbb{R}^n$$
$$v^T (Gu - F) = 0 \qquad \forall v \in \mathbb{R}^n$$

$$\Rightarrow \underline{Gu = F}$$

Exercise 4

(March 19 & 31, 2003)

a) False. Assume $v_1, v_2 \in S$. Then $v_1 + v_2 \notin S$.

b) True. $L : X \to \mathbb{R}$, $L(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 L(v_1) + \alpha_2 L(v_2) \qquad \forall \alpha_1, \alpha_2 \in \mathbb{R}$, $\forall v_1, v_2 \in X$

c) False. $(x,y)_Z = |x||y|$ is not an SPD bilinear form.
Consider $(\alpha_1 x_1 + \alpha_2 x_2, \overline{y}) = |\alpha_1 x_1 + \alpha_2 x_2||\overline{y}|$.
This is <u>not</u> a linear form in the first argument.

d) False. For $w \in H^1(\Omega)$, $|w|_{H^1(\Omega)} = (\int_\Omega |\nabla w|^2 \, \mathrm{d}\Omega)^{\frac{1}{2}}$.
If $w = 1$, $|w|_{H^1(\Omega)} = 0$.

e) $f(x) = x^{3/4}$.

$$\|f\|^2_{L^2(\Omega)} = \int_0^1 (x^{3/4})^2 \, \mathrm{d}x = \int_0^1 x^{3/2} \, \mathrm{d}x = \frac{2}{5} x^{5/2}|_0^1 = \frac{2}{5} < \infty$$
$$\|f\|^2_{H^1(\Omega)} = \int_0^1 (f_x^2 + f^2) \, \mathrm{d}x = \int_0^1 f_x^2 \, \mathrm{d}x + \|f\|^2_{L^2(\Omega)}$$

$$f_x = \frac{3}{4}x^{-1/4} \Rightarrow \int_0^1 f_x^2 \,\mathrm{d}x = \int_0^1 \frac{3}{4}x^{-1/2} \,\mathrm{d}x = \frac{3}{2}x^{1/2}|_0^1 = \frac{3}{2}$$

Hence,

$$\|f\|_{H^1}^2 = \frac{3}{2} + \frac{2}{5} < \infty$$

$$\|f\|_{H^2(\Omega)}^2 = \int_0^1 (f_{xx}^2 + f_x^2 + f^2) \,\mathrm{d}x$$

$$f_{xx} \sim x^{-5/4} \Rightarrow \int_0^1 f_{xx}^2 \not< \infty$$

Conclusion:

$$\underline{f(x) \in H^1(0,1)}$$

f) False.

$$\begin{aligned} w &= e^{-10x} \\ w_x &= -10e^{-10x} = -10w \\ w_{xx} &= 100e^{-10x} = 100w \end{aligned}$$

$$\begin{aligned} |w|_{H^2((0,1))} &= (\int_0^1 w_{xx}^2 \,\mathrm{d}x)^{1/2} = 100\|w\|_{L^2(\Omega)} \\ |w|_{H^1((0,1))} &= (\int_0^1 w_x^2 \,\mathrm{d}x)^{1/2} = 10\|w\|_{L^2(\Omega)} \end{aligned}$$

Exercise 5

(March 19 & 31, 2003)
Given $l \in H^{-1}$, find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = l(v) \qquad \forall v \in H_0^1(\Omega)$$

Stability:

$$\|u\|_{H^1(\Omega)} \leq C\|l\|_{H^{-1}(\Omega)}$$

Assume $\exists u_1$ such that $a(u_1, v) = l(v) \quad \forall v \in H_0^1$
Assume $\exists u_2$ such that $a(u_2, v) = l(v) \quad \forall v \in H_0^1$

By subtracting and using the fact that $a(\cdot,\cdot)$ is bilinear, we find that

$$\begin{aligned} &\exists (u_1 - u_2) \in H_0^1(\Omega) \text{ such that} \\ &a(u_1 - u_2, v) = 0 \quad \forall v \in H_0^1(\Omega) \end{aligned}$$

Now we can use the fact that a is an SPD bilinear form and choose $v = u_1 - u_2$. This gives

$$a(u_1 - u_2, u_1 - u_2) = 0 \quad \Rightarrow \quad \underline{u_1 = u_2}$$

since

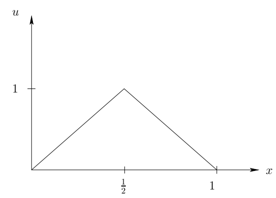
$$a(w, w) = 0 \text{ only when } w = 0$$

Exercise 6

(March 19 & 31, 2003)
Weak form: Find $u \in H_0^1((0,1))$ such that

$$\int_0^1 u_x v_x \, \mathrm{d}x = 4v(\frac{1}{2}) \quad \forall v \in H_0^1(\Omega)$$

By dividing the domain, we have



$$0 < x < \frac{1}{2}: \; u = 1 - 2(\frac{1}{2} - x), u_x = 2$$
$$\frac{1}{2} < x < 1: \; u = 1 - 2(x - \frac{1}{2}), u_x = -2$$

and

$$\begin{aligned}
\int_0^1 u_x v_x \, \mathrm{d}x &= \lim_{\varepsilon \to 0} \int_0^{1/2-\varepsilon} 2v_x \, \mathrm{d}x + \int_{1/2+\varepsilon}^1 (-2) v_x \, \mathrm{d}x \\
&= \lim_{\varepsilon \to 0} \{2[v]_0^{1/2-\varepsilon} - 2[v]_{1/2+\varepsilon}^1\} \\
&= \lim_{\varepsilon \to 0} \{2v(\frac{1}{2} - \varepsilon) - (-2v(\frac{1}{2} + \varepsilon))\} \\
&= \underline{4v(\frac{1}{2})}
\end{aligned}$$

Note that $v \in H_0^1 \Rightarrow v \in C^0(\Omega)$.
Note that $u \notin H^2(\Omega)$, but $u \in H^1(\Omega)$
Note that $f \notin L^2(\Omega)$, but $l(v) \in H^{-1}(\Omega)$.