TMA4220
Numerical solution of partial differential equations
by element methods

Suggested solutions to Problem Set 2[1]

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Exercise 7

(March 19&31, 2003)

$$u_{xxxx} = f \quad \text{in} \quad \Omega = (0,1)$$

$$u(0) = u_x(0) = u(1) = u_x(1) = 0$$

a)

$$\begin{aligned}
\int_0^1 u_{xxxx} v \, \mathrm{d}x &= \int_0^1 f v \, \mathrm{d}x \qquad \forall v \in X \\
\underbrace{[u_{xxx} v]_0^1}_{0} - \int_0^1 u_{xxx} v_x \, \mathrm{d}x &= \int_0^1 f v \, \mathrm{d}x \qquad \forall v \in X \\
-\underbrace{[u_{xx} v_x]_0^1}_{0} + \int_0^1 u_{xx} v_{xx} \, \mathrm{d}x &= \int_0^1 f v \, \mathrm{d}x \qquad \forall v \in X
\end{aligned}$$

Define: $a(w,v) = \int_0^1 w_{xx} v_{xx} \, \mathrm{d}x$.
Note:

$a(w,v)$ is a <u>bilinear</u> form

$a(w,v) = a(v,w)$ — <u>symmetry</u>

$a(w,w) > 0 \quad \begin{matrix} \forall w \in X \\ w \neq 0 \end{matrix}$ — <u>positive definiteness</u>

The problem can be posed as:
Find $u \in X$ such that

$$a(u,v) = l(v) \qquad \forall v \in X$$

with

$$\begin{aligned}
a(w,v) &= \int_0^1 w_{xx} v_{xx} \, \mathrm{d}x \\
l(v) &= \int_0^1 f v \, \mathrm{d}x \\
X &= \{v | v \in H^2(\Omega), v(0) = v(1) = v_x(0) = v_x(1) = 0\}
\end{aligned}$$

See Exercise **3a)** for the rest.

b) H^2 is appropriate.

c) Yes.

$$
\begin{aligned}
l(v) = v_x(\frac{1}{2}) &= \int_0^{1/2} v_{xx}(x)\,\mathrm{d}x && \underline{\text{Note:}}\ v_x(0) = 0 \\
&\leq (\int_0^{1/2} v_{xx}^2\,\mathrm{d}x)^{1/2}(\int_0^{1/2} 1^2\,\mathrm{d}x)^{1/2} && - \quad \text{Cauchy-Schwarz} \\
&= \frac{\sqrt{2}}{2}(\int_0^{1/2} v_{xx}^2\,\mathrm{d}x)^{1/2} \\
&\leq \frac{\sqrt{2}}{2}(\int_0^{1} v_{xx}^2\,\mathrm{d}x)^{1/2} \\
&= \frac{\sqrt{2}}{2}|v|_{H^2(\Omega)} \\
&\leq \frac{\sqrt{2}}{2}\|v\|_{H^2(\Omega)}
\end{aligned}
$$

In fact, we have shown that

$$
|l(v)| \leq C\|v\|_{H^2(\Omega)} \qquad \forall v \in H_0^2(\Omega)
$$

where

$$
H_0^2(\Omega) = \{v \in H^2(\Omega) \mid v(0) = v(1) = v_x(0) = v_x(1) = 0\}
$$

Exercise 8

(March 19&31, 2003)

$$
\begin{aligned}
-u_{xx} &= 1, \qquad u(0) = 0, \qquad u_x(1) = 1 \\
-u_x &= x + A \\
-u &= \frac{1}{2}x^2 + Ax + B \\
u(0) = 0 &\Rightarrow B = 0 \\
u_x(1) = 1 &\Rightarrow 1 + A = -1 \Rightarrow A = -2
\end{aligned}
$$

Hence,

$$
\underline{u = \frac{1}{2}x(4 - x)}
$$

$$
\begin{aligned}
\delta J_v(u) &= \int_0^1 u_x v_x \,\mathrm{d}x - \int_0^1 1\cdot v\,\mathrm{d}x - v(1)\\
&= [u_x v]_0^1 - \int_0^1 u_{xx} v\,\mathrm{d}x - \int_0^1 v\,\mathrm{d}x - v(1)\\
&= u_x(1)v(1) - u_x(0)\underset{0}{v(0)} + \int_0^1 \underbrace{(-u_{xx}-1)}_{=0} v\,\mathrm{d}x - v(1)\\
&= \underbrace{(u_x(1)-1)}_{=0} v(1)\\
&= \underline{0}
\end{aligned}
$$

Exercise 9

(March 19&31, 2003)

a)

$$
\begin{aligned}
\delta J_v(u) &= \int_0^{1/2} \kappa^L u_x^L v_x \,\mathrm{d}x + \int_{1/2}^1 \kappa^R u_x^R v_x \,\mathrm{d}x - \int_0^{1/2} f^L v\,\mathrm{d}x - \int_{1/2}^1 f^R v\,\mathrm{d}x\\
&= [\kappa^L u_x^L v]_0^{1/2} + [\kappa^R u_x^R v]_{1/2}^1 - \int_0^{1/2} \kappa^L u_{xx}^L v\,\mathrm{d}x - \int_{1/2}^1 \kappa^R u_{xx}^R v\,\mathrm{d}x - \int_0^{1/2} f^L v\,\mathrm{d}x - \int_{1/2}^1 f^R v\,\mathrm{d}x\\
&= [\kappa^L u_x^L(\frac{1}{2}) - \kappa^R u_x^R(\frac{1}{2})]\, v(\frac{1}{2}) + \int_0^{1/2} (-\kappa^L u_{xx}^L - f^L) v\,\mathrm{d}x + \int_{1/2}^1 (-\kappa^R u_{xx}^R - f^R) v\,\mathrm{d}x\\
&= \underline{0} \qquad \forall v \in H_0^1(v)
\end{aligned}
$$

a is an SPD bilinear form. Hence,

$$
J(u+v) = J(u) + \frac{1}{2} a(v,v) > J(u) \qquad \begin{array}{c}\forall v \in H_0^1(\Omega)\\ v \neq 0\end{array}
$$

b)

Imposed by $X = H_0^1(\Omega)$: $\begin{cases} u^L(0) = 0 & : \text{ essential} \\ u^R(1) = 0 & : \text{ essential} \end{cases}$

Imposed by J : $\left\{ -\kappa^L u_x^L(\tfrac{1}{2}) = -\kappa^R u_x^R(\tfrac{1}{2}) : \text{ natural} \right.$

$(v(\frac{1}{2})$ unrestricted)

c) $u \in H^1(\Omega)$.

Exercise 10

(March 19&31, 2003)

$$
\begin{aligned}
-\nabla^2 u &= f && \text{in } \Omega \\
u &= 0 && \text{on } \Gamma^D \\
-\frac{\partial u}{\partial n} &= h_c u && \text{on } \Gamma^R \qquad (\overline{\Gamma} = \overline{\Gamma}^D \cup \overline{\Gamma}^R) \\
(h_c &> 0)
\end{aligned}
$$

a)

$$
\begin{aligned}
\int_\Omega -\nabla^2 u v \,\mathrm{d}\Omega &= \int_\Omega f v \,\mathrm{d}\Omega \qquad \forall v \in X \\
\int_\Gamma -\nabla u \cdot \underline{\hat{n}}\; v \,\mathrm{d}S + \int_\Omega \nabla u \nabla v \,\mathrm{d}\Omega &= \int_\Omega f v \,\mathrm{d}\Omega \qquad \forall v \in X
\end{aligned}
$$

Here,

$$
\begin{aligned}
\int_\Gamma -\nabla u \cdot \underline{\hat{n}}\; v \,\mathrm{d}S &= \int_\Gamma -\frac{\partial u}{\partial n} v \,\mathrm{d}S = -\int_{\Gamma^D} \frac{\partial u}{\partial n} \cancelto{0}{v} \,\mathrm{d}S - \int_{\Gamma^R} \frac{\partial u}{\partial n} v \,\mathrm{d}S \\
&= \int_{\Gamma^R} h_c u v \,\mathrm{d}S
\end{aligned}
$$

Hence,

$$
\int_\Omega \nabla u \nabla v \,\mathrm{d}\Omega + \int_{\Gamma^R} h_c u v \,\mathrm{d}S = \int_\Omega f v \,\mathrm{d}\Omega \qquad \forall v \in X
$$

Define:

$$
\begin{aligned}
a(w, v) &= \int_\Omega \nabla w \nabla v \,\mathrm{d}\Omega + \int_{\Gamma^R} h_c w v \,\mathrm{d}S \\
l(v) &= \int_\Omega f v \,\mathrm{d}\Omega
\end{aligned}
$$

Weak formulation: Find $u \in X$ such that

$$
a(w, v) = l(v) \qquad \forall v \in X
$$

a is an SPD bilinear form:

i)

$$
a(\alpha_1 w_1 + \alpha_2 w_2, \overline{v}) = \alpha_1 a(w_1, \overline{v}) + \alpha_2 a(w_2, \overline{v}) \qquad \begin{array}{c} \forall \alpha_1, \alpha_2 \in \mathbb{R} \\ \forall w_1, w_2 \in X \end{array}
$$

$$
a(\overline{w}, \alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 a(\overline{w}, v_1) + \alpha_2 a(\overline{w}, v_2) \qquad \begin{array}{c} \forall \alpha_1, \alpha_2 \in \mathbb{R} \\ \forall v_1, v_2 \in X \end{array}
$$

$\Rightarrow a$ is a bilinear form

$$a : X \times X \to \mathbb{R}$$

ii)

$$a(w,v) = a(v,w) \qquad \Rightarrow \text{symmetry}$$

iii)

$$a(w,w) = \int_\Omega |\nabla w|^2 \, \mathrm{d}\Omega + \int_{\Gamma^R} h_c w^2 \, \mathrm{d}S > 0 \qquad \begin{matrix} \forall w \in X \\ w \neq 0 \end{matrix}$$

Following the results from Exercise **3a)**,

$$u = \arg\min_{w \in X} J(w) = \frac{1}{2} a(w,w) - l(w)$$

and

$$a(u,v) = l(v) \qquad v \in X$$

b)

$$\begin{aligned} u = 0 \quad &\text{on } \Gamma^D : \text{essential (imposed by } X) \\ -\frac{\partial u}{\partial n} = h_c u \ &\text{on } \Gamma^R : \text{natural (imposed by } J \text{ (or a))} \end{aligned}$$

Exercise 11

(March 19&31, 2003)

$$\begin{aligned} u_{xxxx} &= f \text{ in } \Omega = (0,1) \\ u(0) = u_{xx}(0) &= u(1) = u_{xx}(1) = 0 \end{aligned}$$

a)

$$\begin{aligned} \int_0^1 u_{xxxx} v \, \mathrm{d}x &= \int_0^1 f v \, \mathrm{d}x \qquad \forall v \in X \\ \underbrace{[u_{xxx} v]_0^1}_{0} - \int_0^1 u_{xxx} v_x \, \mathrm{d}x &= \int_0^1 f v \, \mathrm{d}x \qquad \forall v \in X \\ [-u_{xx} v_x]_0^1 + \int_0^1 u_{xx} v_{xx} \, \mathrm{d}x &= \int_0^1 f v \, \mathrm{d}x \qquad \forall v \in X \end{aligned}$$

<u>Weak formulation:</u> Find $u \in X$ such that

$$a(u,v) = l(v) \qquad \forall v \in X$$

with

$$\begin{aligned} a(w,v) &= \int_0^1 w_{xx} v_{xx} \, \mathrm{d}x \\ l(v) &= \int_0^1 f v \, \mathrm{d}x \end{aligned}$$

and

$$X = \{v | v \in H^2((0,1)), v(0) = v(1) = 0\}$$

will impose $u_{xx}(0) = u_{xx}(1)$ "naturally".

b)

$$\text{Imposed by } X : \begin{cases} u(0) = 0 \ - \ \text{essential} \\ u(1) = 0 \ - \ \text{essential} \end{cases}$$

$$\text{Imposed by } a(\cdot,\cdot) : \begin{cases} u_{xx}(0) = 0 \ - \ \text{natural} \\ u_{xx}(1) = 0 \ - \ \text{natural} \end{cases}$$

<u>Note:</u> v_x is unrestricted at $x = 0$ and $x = 1$.

Exercise 12

(March 19&31, 2003)

Minimization statement:

$$\begin{aligned} u &= \arg \min_{w \in X^D} J(w) \\ X^D &= \{v \in H^1(\Omega) | v|_{\Gamma^D} = u^D\} \\ X &= \{v \in H^1(\Omega) | v|_{\Gamma^D} = 0\} \end{aligned}$$

Set $w = u + v$ where $u \in X^D$ and $v \in X \Rightarrow \ w \in X^D$.
Then

$$\begin{aligned} J(w) = J(u+v) &= \frac{1}{2} a(u+v, u+v) - l(u+v) \\ &= \frac{1}{2} a(u,u) - l(u) + \frac{1}{2} a(u,v) + \frac{1}{2} a(v,u) - l(v) + \frac{1}{2} a(v,v) \end{aligned}$$

Since $a(u,v) = a(v,u)$ i.e., symmetry

$$\begin{aligned} J(v) &= J(u) + \underbrace{a(u,v) - l(v)}_{\delta J_v(u)} + \frac{1}{2} \underbrace{a(v,v)}_{>0} \qquad \begin{array}{c} \forall v \in X \\ v \neq 0 \\ (a \text{ is SPD}) \end{array} \\ \delta J_v(u) &= \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\Omega - \int_\Omega f v \, \mathrm{d}\Omega \\ &= \int_{\partial\Omega} \nabla u \cdot \underline{\hat{n}} \underset{0}{\cancel{v}} \, \mathrm{d}S - \int_\Omega \nabla^2 u v \, \mathrm{d}\Omega - \int_\Omega f v \, \mathrm{d}\Omega \\ \Rightarrow \delta J_v(u) &= \int_\Omega \underbrace{(-\nabla^2 u - f)}_{=0} v \, \mathrm{d}\Omega \qquad \forall v \in X \\ &= \underline{0} \end{aligned}$$

We have thus shown that, if u satisfies the strong form, then $\delta J_v(u) = 0$ and

$$J(u+v) = J(u) + \frac{1}{2}a(v,v) > J(u) \qquad \begin{array}{c}\forall v \in X \\ v \neq 0\end{array}$$

since a is SPD.
Hence, $u = \arg\min_{w \in X^D} J(w)$

<u>Weak statement:</u> Find $u \in X^D$ such that

$$\begin{aligned}
a(u,v) &= l(v) && \forall v \in X \\
\Rightarrow \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}\Omega &= \int_\Omega f v \,\mathrm{d}\Omega && \forall v \in X \\
\int_{\partial\Omega} \nabla u \cdot \underline{\hat{n}} \underbrace{v}_{0} \,\mathrm{d}\Omega - \int_\Omega \nabla^2 u v \,\mathrm{d}\Omega &= \int_\Omega f v \,\mathrm{d}\Omega && \forall v \in X \\
\int_\Omega (-\nabla^2 u - f) v \,\mathrm{d}\Omega &= 0 && \forall v \in X \\
\Rightarrow -\nabla^2 u &= f && \text{in } \Omega
\end{aligned}$$

Since we search for $u \in X^D$, the strong statement is also satisfied.