TMA4220
Numerical solution of partial differential equations by element methods

Suggested solutions to Problem Set 3[1]

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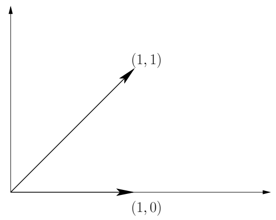
Exercise 1

(April 2, 2003)

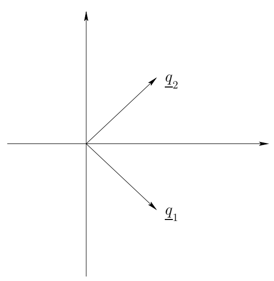
a)



$(1,1),(1,0)$ is a basis for $\mathbb{R}^2$.
It is <u>not</u> an orthogonal basis.
Check: $([1,1],[1,0]) = 1\cdot 1 + 1\cdot 0 = 1 \neq 0$.

b)

$$\underline{q}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \underline{q}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



$$\text{Orthonormal basis} \begin{cases} (\underline{q}_1, \underline{q}_2) & = 0 \\ \|\underline{q}_1\| & = 1 \\ \|\underline{q}_2\| & = 1 \end{cases}$$

Exercise 2

(April 2, 2003)

$Y = \mathbb{P}_2([-1,1]) = \text{span}\{1, x, x^2\}$
$(y,z)_Y = \int_{-1}^{1} yz \,\mathrm{d}x, \quad \forall y,z \in Y$

a)

<u>Observe</u>: $(1,x)_Y = \int_{-1}^{1} 1\cdot x\,\mathrm{d}x = 0$ (orthogonality).
Set $q = a + bx + cx^2, \qquad a,b,c \in \mathbb{R}$.

$$
\begin{aligned}
(1, q) &= \int_{-1}^{1} (a + bx + cx^2)\,\mathrm{d}x = [ax + \frac{1}{2}bx^2 + \frac{1}{3}cx^3]_{-1}^{1} \\
&= 2a + \frac{2}{3}c \\
(x, q) &= \int_{-1}^{1} x(a + bx + cx^2)\,\mathrm{d}x = \int_{-1}^{1} (ax + bx^2 + cx^3)\,\mathrm{d}x \\
&= [\frac{1}{2}ax^2 + \frac{1}{3}bx^3 + \frac{1}{4}cx^4]_{-1}^{1} \\
&= \frac{2}{3}b
\end{aligned}
$$

Orthogonality:

$$
\begin{aligned}
(1, q) &= 2a + \frac{2}{3}c = 0 \Rightarrow a = -\frac{1}{3}c \\
(x, q) &= \frac{2}{3}b = 0 \qquad \Rightarrow b = 0
\end{aligned}
$$

Choose $c = 1$ $\Rightarrow$ $\underline{q = x^2 - \frac{1}{3}}$.
$\{1, x, x^2 - \frac{1}{3}\}$ represent an orthogonal basis.

b) The Legendre polynomials

$$
\begin{aligned}
L_0(x) &= 1 \\
L_1(x) &= x \\
L_2(x) &= \frac{1}{2}(3x^2 - 1) \\
L_3(x) &= \dots \\
&\vdots
\end{aligned}
$$

Exercise 3

(April 2, 2003)

$$
\begin{aligned}
\underline{A}_h &\in R^{n \times n} \\
A_{h_{ij}} &= a(\varphi_i, \varphi_j) = \int_\Omega \frac{\mathrm{d}\varphi_i}{\mathrm{d}x} \frac{\mathrm{d}\varphi_j}{\mathrm{d}x}\,\mathrm{d}x \\
A_{h_{ij}} &= a(\varphi_i, \varphi_j) = a(\varphi_j, \varphi_i) = A_{h_{ji}} \quad - \quad \underline{\text{symmetry}}
\end{aligned}
$$

$$
\begin{aligned}
\underline{v}^T \underline{A}_h \underline{v} &= \sum_{i=1}^{n}\sum_{j=1}^{n} v_i A_{h_{ij}} v_j \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} v_i \Big(\int_\Omega \frac{\mathrm{d}\varphi_i}{\mathrm{d}x}\frac{\mathrm{d}\varphi_j}{\mathrm{d}x}\,\mathrm{d}x\Big) v_j \\
&= \sum_{i=1}^{n} v_i \int_\Omega \frac{\mathrm{d}\varphi_i}{\mathrm{d}x}\Big(\sum_{j=1}^{n} v_j \frac{\mathrm{d}\varphi_j}{\mathrm{d}x}\Big)\,\mathrm{d}x
\end{aligned}
$$

Note that any $v_h \in X_h \subset H_0^1(\Omega)$ can be expressed as:

$$
v_h(x) = \sum_{j=1}^{n} v_j \varphi_j(x)
$$

$$
\begin{aligned}
\Rightarrow \underline{v}^T \underline{A}_h \underline{v} &= \int_\Omega \Big(\sum_{i=1}^{n} v_i \frac{\mathrm{d}\varphi_i}{\mathrm{d}x}\Big)\Big(\sum_{j=1}^{n} v_j \frac{\mathrm{d}\varphi_j}{\mathrm{d}x}\Big)\,\mathrm{d}x \\
&= \int_\Omega \frac{\mathrm{d}v_h}{\mathrm{d}x}\cdot\frac{\mathrm{d}v_h}{\mathrm{d}x}\,\mathrm{d}x = \int_\Omega \Big(\frac{\mathrm{d}v_h}{\mathrm{d}x}\Big)^2\,\mathrm{d}x > 0 \qquad \begin{pmatrix} & \forall v_h & \neq 0 \\ \Rightarrow & \forall \underline{v} & \neq 0 \end{pmatrix}
\end{aligned}
$$

Hence, $\underline{A}_h$ is <u>symmetric</u> and <u>positive-definite</u> (i.e., SPD).

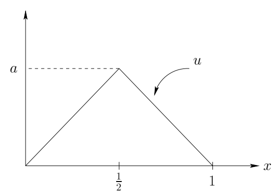
Exercise 4

(April 2, 2003)

[See Exercise 6 from Problem set 1 and Exercise 9 from Problem set 2]
a)

$$
\begin{aligned}
-u^L_{xx} &= 0, \quad 0 < x < \frac{1}{2} \quad &\Rightarrow u^L(x) \text{ is } \underline{\text{linear}} \\
-2u^R_{xx} &= 0, \quad \frac{1}{2} < x < 1 \quad &\Rightarrow u^R(x) \text{ is } \underline{\text{linear}}
\end{aligned}
$$

$$
\left.\begin{aligned} u^L(0) &= u^R(1) &= 0 \\ u^L(\tfrac{1}{2}) &= u^R(\tfrac{1}{2}) & \end{aligned}\right\} \Rightarrow
$$



$$
u^L_x = \frac{a}{1/2} = 2a, \qquad u^R_x = -2a
$$

Now:

$$-u_x^L(\frac{1}{2}) + 1 = -2u_x^R(\frac{1}{2})$$
$$\Rightarrow -2a + 1 = -2(-2a)$$
$$-2a + 1 = 4a$$
$$6a = 1 \quad \Rightarrow \quad \underline{a = \frac{1}{6}}$$

Hence $u = \frac{1}{6} - \frac{1}{6}|x - \frac{1}{2}|, \qquad 0 \leq x \leq 1.$
Note that the flux is discontinous at $x = \frac{1}{2}$. Expect a "heat source" $\propto \delta(x - \frac{1}{2})$.

<u>Weak formulation:</u>
Find $u \in X = H_0^1(\Omega)$ such that

$$\int_0^{1/2} u_x^L v_x \,\mathrm{d}x + \int_{1/2}^1 2u_x^R v_x \,\mathrm{d}x = c \cdot v(\frac{1}{2}) \qquad \forall\, v \in X$$

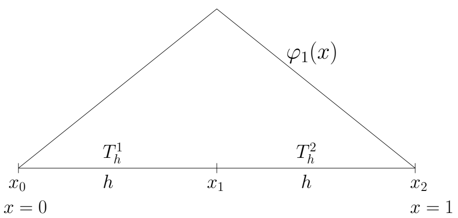
where $c \in \mathbb{R}$.
Here, $u_x^L = 2a = \frac{2}{6} = \frac{1}{3}, \qquad u_x^R = -\frac{1}{3}$

$$\Rightarrow \int_0^{1/2} \frac{1}{3} v_x \,\mathrm{d}x + \int_{1/2}^1 (-\frac{2}{3}) v_x \,\mathrm{d}x = c \cdot v(\frac{1}{2})$$
$$\lim_{\epsilon \to 0} \left\{ [\frac{1}{3}v]_0^{1/2-\epsilon} + [-\frac{2}{3}v]_{1/2+\epsilon}^1 \right\} = c \cdot v(\frac{1}{2})$$
$$\lim_{\epsilon \to 0} \left\{ \frac{1}{3}v(\frac{1}{2} - \epsilon) + \frac{2}{3}v(\frac{1}{2} + \epsilon) \right\} = c \cdot v(\frac{1}{2})$$
$$v(\frac{1}{2}) = c \cdot v(\frac{1}{2})$$
$$\underline{c = 1}$$

Hence, the weak formulation reads:
Find $u \in X = H_0^1((0,1))$ such that

$$\int_0^{1/2} u_x v_x \,\mathrm{d}x + \int_{1/2}^1 2u_x v_x \,\mathrm{d}x = v(\frac{1}{2}) \qquad \forall v \in X$$
$$\left(= \int_0^1 \delta(x - \frac{1}{2}) v(x) \,\mathrm{d}x \right)$$

b)



$$X_h = \mathrm{span}\{\varphi_1\}$$
$$h = \frac{1}{2}$$

$$
\begin{aligned}
A_{h_{11}} &= a(\varphi_1, \varphi_1) \\
&= \int_{T_h^1} \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\,\mathrm{d}x + \int_{T_h^2} 2\frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\,\mathrm{d}x \\
&= \frac{1}{h}\cdot\frac{1}{h}\cdot h + 2\cdot(-\frac{1}{h})(-\frac{1}{h})\cdot h \\
&= \frac{3}{h} = \frac{3}{1/2} = 6
\end{aligned}
$$

$$
F_{h_1} = \int_\Omega f\varphi_1\,\mathrm{d}x = \int_0^1 \delta(x-\frac{1}{2})\varphi_1(x)\,\mathrm{d}x = \varphi_1(\frac{1}{2}) = 1
$$

$$
\begin{aligned}
A_{h_{11}} u_1 &= F_{h_1} \\
6\cdot u_1 &= 1 \Rightarrow \underline{u_1 = \frac{1}{6}} \\
\underline{u_h(x)} &= u_1\varphi_1(x) = \frac{1}{6}\varphi_1 = \underline{\frac{1}{6} - \frac{1}{6}|x-\frac{1}{2}|}
\end{aligned}
$$

c)

We see that $u_h = u$ (exactly).
Recall that

$$
\begin{aligned}
u &= \arg\min_{v\in X} J(v) \\
u_h &= \arg\min_{v\in X_h} J(v)
\end{aligned}
$$

Since, in fact, $u \in X_h$, $u_h = u$.

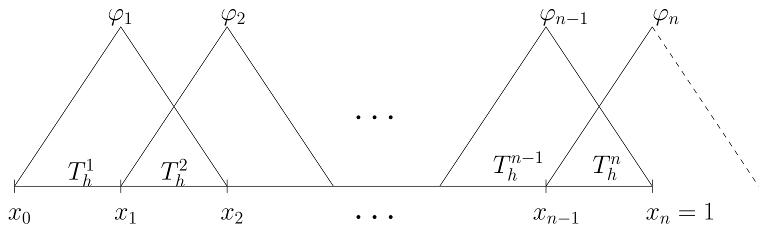
Exercise 5

(April 2, 2003)

$$
\begin{aligned}
-u_{xx} &= f \qquad && \text{in } \Omega = (0,1) \\
u(0) &= 0, \qquad && u_x(1) = g
\end{aligned}
$$

a)

$$
\begin{aligned}
X &= \{v \in H^1(\Omega) | v(0) = 0\} \\
X_h &= \{v \in X | v|_{T_h^k} \in \mathbb{P}_1(T_h^k), k = 1,\dots,K\} \\
X_h &= \operatorname{span}\{\varphi_1, \varphi_2, \dots, \varphi_n|_{T_h^n}\} \qquad (K = n)
\end{aligned}
$$



b)

<u>Weak statement:</u> Find $u \in X$ such that

$$a(u,v) = l(v) \qquad \forall\ v \in X$$

where

$$\begin{aligned} a(u,v) &= \int_0^1 u_x v_x \,\mathrm{d}x \\ l(v) &= \int_0^1 f v \,\mathrm{d}x + g v(1) \end{aligned}$$

<u>Discrete problem:</u> Find $u_h \in X_h$ such that

$$a(u_h, v) = l(v) \qquad \forall\ v \in X_h$$

Here

$$\begin{aligned} v_h(x) &= \sum_{i=1}^{n} v_i \varphi_i(x) \qquad 0 \leq x \leq 1 \\ u_h(x) &= \sum_{j=1}^{n} u_{h_j} \varphi_j(x) \qquad 0 \leq x \leq 1 \\ \Rightarrow \underline{A}_h \underline{u}_h &= \underline{F}_h \end{aligned}$$

where

$$
\begin{aligned}
A_{h_{ij}} &= a(\varphi_i, \varphi_j) = \int_{T_h^i} \frac{\mathrm{d}\varphi_i}{\mathrm{d}x}\frac{\mathrm{d}\varphi_j}{\mathrm{d}x}\,\mathrm{d}x + \underbrace{\int_{T_h^{i+1}} \frac{\mathrm{d}\varphi_i}{\mathrm{d}x}\frac{\mathrm{d}\varphi_j}{\mathrm{d}x}\,\mathrm{d}x}_{\text{for } i=1,2,\ldots,n-1} \\
A_{h_{ii}} &= \frac{1}{h}\cdot\frac{1}{h}\cdot h + (-\frac{1}{h})\cdot(-\frac{1}{h})\cdot h = \frac{2}{h} & i = 2,3,\ldots,n-1 \\
A_{h_{i,i-1}} &= -\frac{1}{h} & i = 2,3,\ldots,n-1 \\
A_{h_{i,i+1}} &= -\frac{1}{h} & i = 2,3,\ldots,n-1 \\
A_{h_{11}} &= \frac{2}{h}, \quad A_{h_{12}} = -\frac{1}{h} \\
A_{h_{nn}} &= \int_{T_h^n} \frac{\mathrm{d}\varphi_n}{\mathrm{d}x}\frac{\mathrm{d}\varphi_n}{\mathrm{d}x}\,\mathrm{d}x = \frac{1}{h}\cdot\frac{1}{h}\cdot h = \frac{1}{h} \\
A_{h_{n,n-1}} &= \int_{T_h^n} \frac{\mathrm{d}\varphi_n}{\mathrm{d}x}\frac{\mathrm{d}\varphi_{n-1}}{\mathrm{d}x}\,\mathrm{d}x = \frac{1}{h}\cdot(-\frac{1}{h})\cdot h = -\frac{1}{h} \\
F_{h_i} &= l(\varphi_i) = \int_{T_h^i} f\varphi_i\,\mathrm{d}x + \int_{T_h^{i+1}} f\varphi_i\,\mathrm{d}x, & i = 1,2,\ldots,n-1 \\
F_{h_n} &= l(\varphi_n) = \int_{T_h^n} f\varphi_n\,\mathrm{d}x + g\underbrace{\varphi_n(1)}_{=1}
\end{aligned}
$$

Hence, $\underline{u}_h \in \mathbb{R}^n$ satisfies

$$
\frac{1}{h}\begin{bmatrix}
2 & -1 & & & & & \\
-1 & 2 & -1 & & & 0 & \\
 & -1 & 2 & -1 & & & \\
 & & & \ddots & & & \\
 & & & & \ddots & & \\
 & 0 & & & & 2 & -1 \\
 & & & & & -1 & 1
\end{bmatrix}
\begin{bmatrix} u_{h_1} \\ u_{h_2} \\ u_{h_3} \\ \vdots \\ \vdots \\ u_{h_{n-1}} \\ u_{h_n} \end{bmatrix}
=
\begin{bmatrix} F_{h_1} \\ F_{h_2} \\ F_{h_3} \\ \vdots \\ \vdots \\ F_{h_{n-1}} \\ F_{h_n} \end{bmatrix}
$$

c)

Note that the n'th equation reads

$$
\frac{u_{h_n} - u_{h_{n-1}}}{h} = \int_{T_h^n} f\varphi_n\,\mathrm{d}x + g
$$

A one-sided finite difference scheme would read

$$
\frac{u_{h_n} - u_{h_{n-1}}}{h} = g
$$

In the finite element context, the last equation represent a "mix" between satisfying the differential equation <u>and</u> satisfying the boundary condition. Note that the term $\int_{T_h^n} f\varphi_n\,\mathrm{d}x = \mathcal{O}(h) \underset{h\to 0}{\to} 0$ (if $f \in L^2(\Omega)$).