TMA4220
Numerical solution of partial differential equations
by element methods

Suggested solutions to Problem Set 4[1]

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Exercise 6

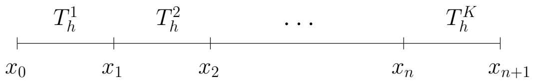
(April 2, 2003)

$$X_h = \{v \in X | v|_{T_h} \in \mathbb{P}_2(T_h), \forall\, T_h \in \mathcal{T}_h\}$$

Here $X = H_0^1(\Omega)$ (Dirichlet problem)

a)

$$v|_{T_h^k} \in \mathbb{P}_2(T_h^k) \Rightarrow v(x)|_{T_h^k} = a_k + b_k x + c_k x^2, \quad k = 1, \ldots, K$$



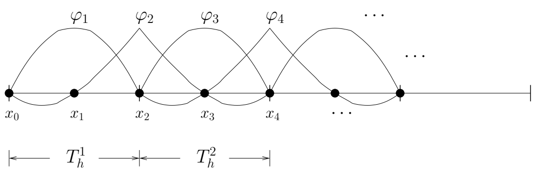
$\underline{K = n+1}$

Counting argument:

$$\dim(X_h) = \underset{\substack{\uparrow \\ \text{degrees-of-freedom} \\ \text{associated with} \\ \text{a polynomial of} \\ \text{degree 2}}}{3}\ \overbrace{(n+1)}^{\substack{\text{number of} \\ \text{elements}}} \;-\; \underset{\substack{\uparrow \\ \text{boundary} \\ \text{conditions}}}{2} \;-\; \overset{\substack{\text{internal} \\ \text{continuity} \\ \text{conditions} \\ v \in H_0^1(\Omega) \\ \downarrow}}{n} \;= 2n+1$$

$\underline{\dim(X_h) = 2n+1}$.

b)



$$\varphi_i \in X_h$$
$$\varphi_i(x_j) = \delta_{ij}$$

c)

Matrix structure (using a natural ordering):

$$
A_h = \begin{bmatrix}
x & x & & & & & & \\
x & x & x & x & & & & \\
 & x & x & x & & & & \\
 & x & x & x & x & x & & \\
 & & & x & x & x & & \\
 & & & x & x & x & & \\
 & & & & & & \ddots & \\
 & & & & & & & \\
 & & & & & & &
\end{bmatrix}
$$

Five diagonals with non-zero elements. (Recall that linear elements resulted in A_h being tridiagonal).
Variable bandwidth due to different kinds of basis functions.
We will return to this exercise later in the course.

Exercise 7

(April 2, 2003)

$$u_{xxxx} = f \text{ in } \Omega = (0,1)$$

From Exercise 7 & 11 of Problem set 2, we concluded:

$$
\begin{aligned}
u \in & H^2(\Omega) \Rightarrow u \in C^1(\Omega) \\
&\Rightarrow u \text{ is continuous} \\
&\quad u_x \text{ is continuous}
\end{aligned}
$$

If

$$
\begin{aligned}
u_h \in X_h \subset H^2(\Omega) \ \Rightarrow \ & u_h \text{ is continuous} \\
& (u_h)_x \text{ is continuous}
\end{aligned}
$$

Consider linear elements:

$K = n + 1$

If $u_h|_{T_h^k} \in \mathbb{P}_1(T_h^k) \ \Rightarrow \ u_h|_{T_h^k} = a_k + b_k x$.
Let m be the number of essential b.c.'s. Then, the number of degrees-of-freedom is:

$$N = 2(n+1) - m - 2n = 2 - m.$$

In Exercise 7 (Problem set 2), $m = 4 \Rightarrow N = -2$.
In Exercise 11 (Problem set 2), $m = 2 \Rightarrow N = 0$.

Conclusion: If we insist that $u_h \in X_h \subset X$ (i.e., conforming approximation), we cannot use linear elements. With linear elements, we do not have any degrees-of-freedom left to approximate the solution.

Exercise 8

(April 2, 2003)

$$\begin{aligned} -u_{xx} + \gamma^2 u &= f \text{ in } \Omega = (0,1) \\ u(0) = u(1) &= 0 \end{aligned}$$

a)

$$\begin{aligned} &\int_0^1 (-u_{xx} + \gamma^2 u) v \,\mathrm{d}x = \int_0^1 f v \,\mathrm{d}x \qquad \forall v \in X \\ \Rightarrow &\int_0^1 (u_x v_x + \gamma^2 u v) \,\mathrm{d}x = \int_0^1 f v \,\mathrm{d}x \qquad \forall v \in X \end{aligned}$$

Weak form: Find $u \in X$ such that

$$a(u,v) = l(v) \qquad \forall v \in X$$

where

$$\begin{aligned} a(w,v) &= \int_0^1 (w_x v_x + \gamma^2 w v) \,\mathrm{d}x \\ l(v) &= \int_0^1 f v \,\mathrm{d}x \end{aligned}$$

$$\underline{\text{SPD}} \begin{cases} a(w,v) = a(v,w) \\ a(w,w) = \int_0^1 (w_x^2 + \gamma^2 w^2) \,\mathrm{d}x > 0 & \begin{matrix} \forall w \in X \\ w \neq 0 \end{matrix} \end{cases}$$

$$\begin{aligned} X &= \{ v \in H^1(\Omega) | v(0) = v(1) = 0 \} \\ J(w) &= \frac{1}{2} a(w,w) - l(w) \\ u &= \arg\min_{w \in X} J(w) \qquad \text{-minimization statement} \end{aligned}$$

b)

Discrete problem: Find $u_h \in X_h \subset X$ such that

$$a(u_h, v) = l(v) \qquad \forall v \in X_h$$

Let $X_h = \text{span}\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ be a nodal basis.

$$\begin{aligned} \Rightarrow \;\; u_h &= \sum_{j=1}^{n} u_{h_j} \varphi_j(x) \\ v &= \sum_{i=1}^{n} \varphi_i(x) \qquad \forall v \in X_h \end{aligned}$$

The discrete problem then becomes: Find $\underline{u}_h \in \mathbb{R}^n$ such that

$$\begin{aligned} a(\sum_{j=1}^{n} u_{h_j}\varphi_j, \sum_{i=1}^{n} v_i \varphi_i) &= l(\sum_{i=1}^{n} v_i \varphi_i) \qquad \underline{v} \in \mathbb{R}^n \\ \sum_{i=1}^{n}\sum_{j=1}^{n} v_i a(\varphi_i, \varphi_j) u_{h_j} &= \sum_{i=1}^{n} v_i l(\varphi_i) \\ \rightarrow \underline{A}_h \underline{u}_h &= \underline{F}_h \end{aligned}$$

where

$$A_{h_{ij}} = a(\varphi_i, \varphi_j) = \underbrace{\int_0^1 \frac{\mathrm{d}\varphi_i}{\mathrm{d}x}\frac{\mathrm{d}\varphi_j}{\mathrm{d}x}\,\mathrm{d}x}_{A_{h_{ij}}^{\text{Laplacian}}} + \underbrace{\gamma^2 \int_0^1 \varphi_i \varphi_j \,\mathrm{d}x}_{M_{h_{ij}}}$$

Hence:

$$\underline{A}_h \equiv \underline{A}_h^{\text{Helmholtz}} = \underline{A}_h^{\text{Laplacian}} + \gamma^2 \underline{M}_h$$

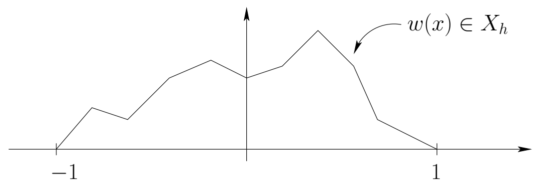
Exercise 9

(April 2, 2003)

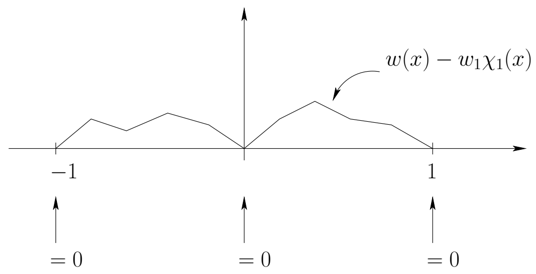
a)

$$\begin{aligned} w(x) &= \sum_{i=1}^{n} w_i \chi_i(x) \\ X_h &= \{v \in X = H_0^1(\Omega) | v|_{T_h^k} \in \mathbb{P}_1(T_h^k), k = 1, \dots, K\} \end{aligned}$$

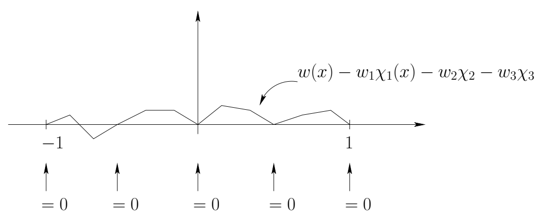
By construction, each $\chi_i = \chi_{\text{ind}(m,j)} = \psi_{m,j} \in X_h$. Since X_h is a linear space, $w(x) \in X_h$.
Consider $w(x)$



<u>Note</u>: $w(0) = w_1\chi_1(0) = w_1$



$$\underline{\text{Note}}: \ (w - w_1\chi_1)(-\frac{1}{2}) = w_2\chi_2(-\frac{1}{2}) = w_2$$
$$(w - w_1\chi_1)(\frac{1}{2}) = w_3\chi_3(+\frac{1}{2}) = w_3$$



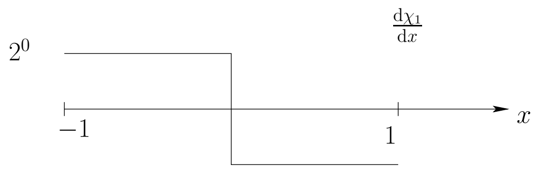
In this way, we can determine all the w_i in a unique fashion.

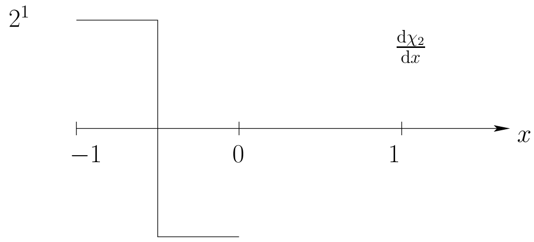
b)

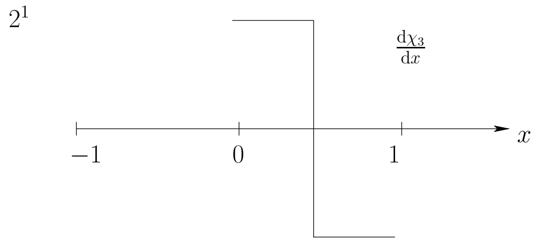
$$-u_{xx} = f$$
$$u(-1) = u(1) = 0$$

Find $u_h \in X_h = \text{span}\{\chi_i, i = 1, \dots, n\}$ such that

$$
\begin{aligned}
& a(u_h, v) = l(v) && \forall v \in X_h \\
\Rightarrow\ & \underline{A}_h \underline{u}_h = \underline{F}_h \\
& (A_h)_{ij} = \int_{-1}^{1} \frac{\mathrm{d}\chi_i}{\mathrm{d}x} \frac{\mathrm{d}\chi_j}{\mathrm{d}x} \,\mathrm{d}x && 1 \le i, j \le n
\end{aligned}
$$



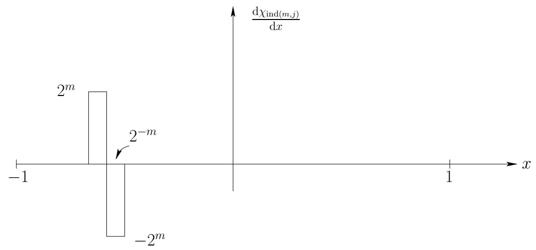




etc. . . .

Observe that $\int_{-1}^{1} \frac{\mathrm{d}\chi_i}{\mathrm{d}x} \frac{\mathrm{d}\chi_j}{\mathrm{d}x} \,\mathrm{d}x = 0, \quad i \neq j$. Hence, $\underline{A}_h$ is diagonal $\Rightarrow$ trivial to solve for $\underline{u}_h$.
For each level $m = 0, \dots, L$,

$$
\left(\frac{\mathrm{d}\chi_{\mathrm{ind}(m,j)}}{\mathrm{d}x}\right)^2 = (2^m)^2, \qquad j = 1, \dots, 2^m
$$



Hence,

$$(A_h)_{\mathrm{ind}(m,j)\mathrm{ind}(m,j)} = (2^m)^2 \cdot 2 \cdot 2^{-m} = \underline{2^{m+1}} \qquad \begin{array}{r} j = 1, \dots, 2^m \\ m = 0, \dots, L \end{array}$$

c)

$$\begin{aligned} -u_{xx} + u &= f \text{ in } \Omega \\ u(-1) = u(1) &= 0 \\ \Rightarrow \ \underline{A}_h \underline{u}_h &= \underline{F}_h \end{aligned}$$

where

$$(A_h)_{ij} = \underbrace{\int_{-1}^{1} \frac{\mathrm{d}\chi_i}{\mathrm{d}x} \frac{\mathrm{d}\chi_j}{\mathrm{d}x} \,\mathrm{d}x}_{\text{diagonal}} + \underbrace{\int_{-1}^{1} \chi_i \chi_j \,\mathrm{d}x}_{\substack{\text{quite dense} \\ \text{e.g., } \int_{-1}^{1} \chi_1 \chi_j \,\mathrm{d}x \neq 0 \\ \forall j = 1, \dots, n}}$$

$\Rightarrow$ expensive to solve for $\underline{u}_h$.

Exercise 1

(April 7 and 9, 2003)

a)

Let Y and $Z \subset Y$ be Hilbert spaces. Let $y \in Y$. Then $\Pi y \in Z$ such that

$$(\Pi y, v)_Y = (y, v)_Y \qquad \forall v \in Z$$

Choose $v = \Pi y \in Z \subset Y$.

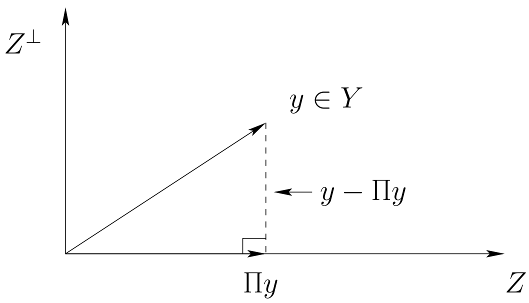
$$
\begin{aligned}
(\Pi y, \Pi y)_Y &= (y, \Pi y)_Y \\
\|\Pi y\|_Y^2 &= (y, \Pi y)_Y \\
&\leq \|y\|_Y \|\Pi y\|_Y \qquad \text{(Cauchy-Schwarz)} \\
\Rightarrow \ \underline{\|\Pi y\|_Y \leq \|y\|_Y}
\end{aligned}
$$

We know that

$$\|y - \Pi y\|_Y \leq \|y - z\|_Y \qquad \forall z \in Z$$

In particular, choose $z = 0 \ \in Z$:
Geometric interpretation:

$$\underline{\|y - \Pi y\|_Y \leq \|y\|_Y}$$



b)

$$y' \in Y, \quad \Pi y' \in Z \subset Y$$

$$(\Pi y', v)_Y = (y', v)_Y \qquad \forall v \in Z$$

In particular, for $y' = \Pi y \in Z$, where $y \in Y$, we get

$$(\Pi(\Pi y), v)_Y = (\Pi y, v)_Y \qquad \forall v \in Z$$

or

$$(\Pi y - \Pi(\Pi y), v)_Y = 0 \qquad \forall v \in Z$$

But

$$
\begin{aligned}
\Pi y &\in Z \\
\Pi(\Pi y) &\in Z
\end{aligned}
$$

Choose $v = \Pi y - \Pi(\Pi y) \in Z$:

$$
\begin{aligned}
\Rightarrow \;\; (\Pi y - \Pi(\Pi y), \Pi y - \Pi(\Pi y))_Y &= 0 \\
\|\Pi y - \Pi(\Pi y)\|_Y^2 &= 0 \\
\Rightarrow \;\; \Pi y - \Pi(\Pi y) &= 0
\end{aligned}
$$

or

$$
\underline{\Pi(\Pi y) = \Pi y} \qquad \forall y \in Y \quad .
$$

Exercise 2

(April 7 and 9, 2003)

$$
\begin{aligned}
(w - I_h w)|_{T_h^k}(x) = &\underset{\substack{\uparrow \\ \text{(since } (w - I_h w)|_{T_h^k}(0) = 0)}}{\int_0^x} (w - I_h w)'|_{T_h^k}(\xi)\,\mathrm{d}\xi \\
&= \int_0^x \Big[\int_{z^k}^{\xi} (w - I_h w)''|_{T_h^k}(\eta)\,\mathrm{d}\eta\Big]\,\mathrm{d}\xi \\
&= \int_0^x \Big[\int_{z^k}^{\xi} w''\,\mathrm{d}\eta\Big]\,\mathrm{d}\xi \\
&\le \int_0^x \big[h \max_{\eta \in T_h^k} |w''(\eta)|\big]\,\mathrm{d}\xi \\
&= h \max_{x \in T_h^k} |w''(x)| \cdot \int_0^x \mathrm{d}\xi \\
&\le h^2 \max_{x \in T_h^k} |w''(x)|
\end{aligned}
$$

$$
\begin{aligned}
\|w - I_h w\|_{L^2(\Omega)}^2 &= \sum_{k=1}^{K} \int_{T_h^k} (w - I_h w)^2|_{T_h^k}\,\mathrm{d}x \\
&\le \frac{1}{h} \cdot h \cdot (h^2 \max_k \max_{x \in T_h^k} |w''|)^2 \\
&= h^4 (\max_{T_h \in \mathcal{T}_h} \max_{x \in T_h} |w''|)^2 \\
\Rightarrow \;\; \underline{\|w - I_h w\|_{L^2(\Omega)}} &\underline{\le h^2 \max_{T_h \in \mathcal{T}_h} \max_{x \in T_h} |w''|}
\end{aligned}
$$

Exercise 3

(April 7 and 9, 2003)

Consider the Poisson problem in $\mathbb{R}^1$:
Find $u \in X = H_0^1(\Omega)$ such that

$$a(u, v) = l(v) \qquad \forall v \in X$$

where

$$\begin{aligned} a(w, v) &= \int_0^1 w_x v_x \, \mathrm{d}x \\ l(v) &= \int_0^1 f v \, \mathrm{d}x \end{aligned}$$

Discrete problem using <u>linear</u> finite elements:
Find $u_h \in\subset X$ such that

$$a(u_h, v) = l(v) \qquad \forall v \in X_h$$

Orthogonality:

$$\underline{a(u - u_h, v) = 0 \qquad \forall v \in X_h}$$

Consider

$$\begin{aligned} a(u - I_h u, v) &= \int_0^1 (u_x - (I_h u)_x) v_x \, \mathrm{d}x \\ &= \sum_k \int_{T_h^k} (u_x - (I_h u)_x) v_x \, \mathrm{d}x \\ &= \sum_k [\underbrace{(u - I_h u)}_{\substack{\uparrow \\ = 0 \\ \text{by definition} \\ \text{of the interpolant}}} v_x]_{\partial T_h^k} - \int_{T_h^k} (u - I_h u) \underbrace{v_{xx}}_{\substack{\uparrow \\ = 0 \\ (v \in X_h)}} \, \mathrm{d}x \end{aligned}$$

$$\begin{aligned} &\Rightarrow \; a(u - I_h u, v) = 0 \qquad \forall v \in X_h \\ &\Rightarrow \underline{u_h = I_h u} \qquad \text{(due to the uniqueness of the solution)} \end{aligned}$$

Note that, since the orthogonality relation is satisfied for both u_h and $I_h u$, the distance $\| u - u_h \|$ and $\| u - I_h u \|$ are both minimized over all elements in X_h, i.e., u_h and $I_h u$ are both the closest we can come to the exact solution measured in the energy norm. Due to the uniqueness of the discrete solution, $u_h = I_h u$. Note that, in general, this result is *not* true in higher space dimensions.

Exercise 4

(April 7 and 9, 2003)

General bound

$$\| e \| = \inf_{w_h \in X_h} \| u - w_h \|$$

where

$$e = u - u_h, \qquad u \in X, \qquad u_h \in X_h \subset X$$
$$\| e \|^2 = a(e,e)$$

If

$$a(w,v) = \int_\Omega (\nabla w \cdot \nabla v + w \cdot v)\, \mathrm{d}\Omega \tag{$*$}$$

then

$$\| e \|^2 \;=\; a(e,e) = \int_\Omega (\nabla e \cdot \nabla e + e \cdot e)\ \mathrm{d}\Omega = \int_\Omega (|\nabla e|^2 + e^2)\ \mathrm{d}\Omega \;=\; \| e \|^2_{H^1(\Omega)}$$

Hence, in this particular case,

$$\|e\|_{H^1(\Omega)} = \inf_{w_h \in X_h} \|u - w_h\|_{H^1(\Omega)}$$

that is, u_h is the H^1 projection of u on X_h.
The SPD bilinear form $(*)$ corresponds to solving the Helmholtz equation

$$-\nabla^2 u + u = f \text{ in } \Omega$$

with

$$u = 0 \text{ on } \partial\Omega \text{ (for example)}$$