TMA4220
Numerical solution of partial differential equations by element methods

Suggested solutions to Problem Set 5[1]

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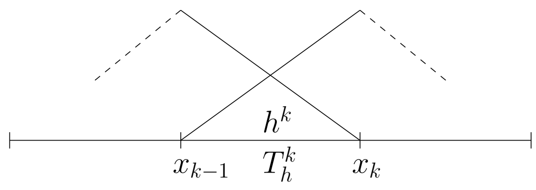
Exercise 5

(April 7 and 9, 2003)



Consider

$$\int_{T_h^k} \frac{\mathrm{d}\varphi_i}{\mathrm{d}x}(x)\frac{\mathrm{d}\varphi_j}{\mathrm{d}x}(x)\,\mathrm{d}x, \qquad \begin{array}{ll} i & = (k-1) \text{ or } k \\ j & = (k-1) \text{ or } k \end{array}$$

$$\begin{aligned}
&= \int_{T_h^k} \frac{\mathrm{d}H_\alpha(\mathcal{F}_k^{-1}(x))}{\mathrm{d}x}\frac{\mathrm{d}H_\beta(\mathcal{F}_k^{-1}(x))}{\mathrm{d}x}\,\mathrm{d}x \qquad \begin{array}{ll} \alpha & = 1 \text{ or } 2 \\ \beta & = 1 \text{ or } 2 \end{array} \\
&= \int_{\hat{T}} \left(\frac{\mathrm{d}H_\alpha(\zeta)}{\mathrm{d}\zeta}\cdot\frac{\mathrm{d}\zeta}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}H_\beta(\zeta)}{\mathrm{d}\zeta}\cdot\frac{\mathrm{d}\zeta}{\mathrm{d}x}\right)\cdot\frac{\mathrm{d}x}{\mathrm{d}\zeta}\,\mathrm{d}\zeta \\
&= \frac{\mathrm{d}\zeta}{\mathrm{d}x}\int_{-1}^{1} \frac{\mathrm{d}H_\alpha(\zeta)}{\mathrm{d}\zeta}\cdot\frac{\mathrm{d}H_\beta(\zeta)}{\mathrm{d}\zeta}\,\mathrm{d}\zeta \\
&= \frac{2}{h^k}\int_{-1}^{1} \frac{\mathrm{d}H_\alpha}{\mathrm{d}\zeta}\frac{\mathrm{d}H_\beta}{\mathrm{d}\zeta}\,\mathrm{d}\zeta \qquad \begin{array}{ll} \alpha & = 1 \text{ or } 2 \\ \beta & = 1 \text{ or } 2 \end{array}
\end{aligned}$$

$$\begin{aligned}
H_1(\zeta) &= \frac{1-\zeta}{2} \quad \Rightarrow \quad \frac{\mathrm{d}H_1}{\mathrm{d}\zeta} = -\frac{1}{2} \\
H_2(\zeta) &= \frac{1+\zeta}{2} \quad \Rightarrow \quad \frac{\mathrm{d}H_2}{\mathrm{d}\zeta} = +\frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \left[\ \frac{2}{h^k}\int_{-1}^{1}\frac{\mathrm{d}H_\alpha}{\mathrm{d}\zeta}\frac{\mathrm{d}H_\beta}{\mathrm{d}\zeta}\,\mathrm{d}\zeta\ \right]_{\alpha,\beta=1}^{2} &= \frac{2}{h^k}\begin{bmatrix} \int_{-1}^{1}\frac{\mathrm{d}H_1}{\mathrm{d}\zeta}\frac{\mathrm{d}H_1}{\mathrm{d}\zeta}\,\mathrm{d}\zeta & \int_{-1}^{1}\frac{\mathrm{d}H_1}{\mathrm{d}\zeta}\frac{\mathrm{d}H_2}{\mathrm{d}\zeta}\,\mathrm{d}\zeta \\ \int_{-1}^{1}\frac{\mathrm{d}H_2}{\mathrm{d}\zeta}\frac{\mathrm{d}H_1}{\mathrm{d}\zeta}\,\mathrm{d}\zeta & \int_{-1}^{1}\frac{\mathrm{d}H_2}{\mathrm{d}\zeta}\frac{\mathrm{d}H_2}{\mathrm{d}\zeta}\,\mathrm{d}\zeta \end{bmatrix} \\
&= \frac{2}{h^k}\begin{bmatrix} (-\frac{1}{2})\cdot(-\frac{1}{2})\cdot 2 & (-\frac{1}{2})\cdot(\frac{1}{2})\cdot 2 \\ (\frac{1}{2})\cdot(-\frac{1}{2})\cdot 2 & (\frac{1}{2})\cdot(\frac{1}{2})\cdot 2 \end{bmatrix} \\
&= \frac{2}{h^k}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \\
&= \frac{1}{h^k}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\
&\equiv \underline{A}^k
\end{aligned}$$

Exercise 6

(April 7 and 9, 2003)

a)

$$\underline{\tilde{M}}_h \in \mathbb{R}^{(n+1)\times(n+1)}$$

$$(\tilde{M}_h)_{ij} = \int_0^1 \varphi_i \varphi_j \,\mathrm{d}x \qquad 0 \le i, j \le n+1$$

$$\tilde{X}_h = \{v \in H^1(\Omega) \quad , \quad v_{|T_h^k} \in \mathbb{P}_1(T_h^k), k = 1, \dots, K\}$$
$$= \mathrm{span}\{\varphi_0, \varphi_1, \varphi_2, \dots, \varphi_n, \varphi_{n+1}\}$$

$$v(x) = \sum_{i=0}^{n+1} \underset{\substack{\uparrow \\ \text{nodal basis: } \underline{v_i = v(x_i)}}}{v_i} \varphi_i(x) \qquad \forall\ v(x) \in \tilde{X}_h$$

In particular:

$$v(x) = 1 \in \tilde{X}_h \quad \Rightarrow \quad \sum_{i=0}^{n+1} \varphi_i(x) = 1$$

Hence

$$\begin{aligned}
\sum_{i=0}^{n+1}\sum_{j=0}^{n+1} \tilde{M}_{h_{ij}} &= \sum_{i=0}^{n+1}\sum_{j=0}^{n+1} \int_0^1 \varphi_i \varphi_j \,\mathrm{d}x \\
&= \int_0^1 (\sum_{i=0}^{n+1} \varphi_i)(\sum_{j=0}^{n+1} \varphi_j) \,\mathrm{d}x \\
&= \int_0^1 1 \cdot 1 \,\mathrm{d}x \\
&= \underline{1} \qquad \text{Q.E.D.}
\end{aligned}$$

b)

$$\int_{T_h^k} \varphi_i \varphi_j \, \mathrm{d}x \qquad \begin{array}{l} i = (k-1) \text{ or } k \\ j = (k-1) \text{ or } k \end{array}$$

$$= \int_{T_h^k} H_\alpha(\mathcal{F}_k^{-1}(x)) H_\beta(\mathcal{F}_k^{-1}(x)) \, \mathrm{d}x \qquad \begin{array}{l} \alpha = 1 \text{ or } 2 \\ \beta = 1 \text{ or } 2 \end{array}$$

$$= \int_{\hat{T}} H_\alpha(\zeta) H_\beta(\zeta) \frac{\mathrm{d}x}{\mathrm{d}\zeta} \, \mathrm{d}\zeta$$

$$= \frac{h^k}{2} \int_{-1}^{1} H_\alpha H_\beta \, \mathrm{d}\zeta$$

$$\underline{M}^k = \frac{h^k}{2} \begin{bmatrix} \int_{-1}^1 H_1 H_1 \, \mathrm{d}\zeta & \int_{-1}^1 H_1 H_2 \, \mathrm{d}\zeta \\ \int_{-1}^1 H_2 H_1 \, \mathrm{d}\zeta & \int_{-1}^1 H_2 H_2 \, \mathrm{d}\zeta \end{bmatrix}$$

For example: $\int_{-1}^1 H_1 H_1 \, \mathrm{d}\zeta = \int_{-1}^1 (\frac{1-\zeta}{2})^2 \, \mathrm{d}\zeta = \frac{1}{4} \int_{-1}^1 (1 - 2\zeta + \zeta^2) \, \mathrm{d}\zeta = \frac{2}{3}$

$$\underline{\Rightarrow \quad \underline{M}^k = \frac{h^k}{2} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}}$$

Exercise 7

(April 7 and 9, 2003)

$$-u_{xx} = f \text{ in } \Omega = (0,1)$$
$$u(0) = u^D, \; u_x(1) = 0$$

Point of departure:

$$\frac{1}{h} \begin{bmatrix} 1 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}_{h_0} \\ \tilde{u}_{h_1} \\ \vdots \\ \vdots \\ \tilde{u}_{h_n} \\ \tilde{u}_{h_{n+1}} \end{bmatrix} = \begin{bmatrix} \tilde{F}_{h_0} \\ \tilde{F}_{h_1} \\ \vdots \\ \vdots \\ \tilde{F}_{h_n} \\ \tilde{F}_{h_{n+1}} \end{bmatrix}$$

<u>Weak form</u>: Find $u_h \in X_h^D$ such that

$$a(u_h, v) = l(v) \qquad \forall v \in X_h$$

$$X_h \text{ requires } v(0) = 0$$
$$X_h^D \text{ requires } v(0) = u^D$$

$X_h \Rightarrow \varphi_0$ not admissable $\rightarrow$ <u>remove</u> RO

$X_h^D \Rightarrow \tilde{u}_{h_0} = u^D \rightarrow$ <u>move</u> $- u^D \times$ CO to the right hands side

Explicit elimination:

$$\frac{1}{h}\begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 1 \end{bmatrix} \begin{bmatrix} u_{h_1} \\ \\ \vdots \\ \vdots \\ \\ u_{h_{n+1}} \end{bmatrix} = \begin{bmatrix} \tilde{F}_{h_1} - u^D \times (-\frac{1}{h}) \\ \tilde{F}_{h_2} \\ \vdots \\ \vdots \\ \\ \tilde{F}_{h_{n+1}} \end{bmatrix}$$

Big-Number approach

Place $\frac{1}{\epsilon}(\epsilon \ll 1)$ on entries $(\tilde{A}_h)_{00}$

Place $(\frac{1}{\epsilon}) \cdot u^D$ on entry $\tilde{F}_{h_0}$.