TMA4220

Numerical solution of partial differential equations by element methods

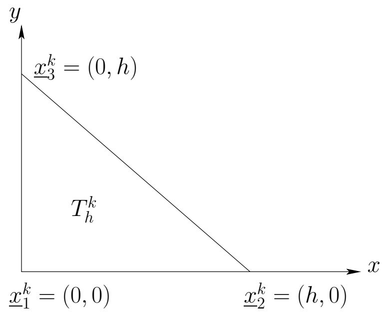
Suggested solutions to Problem Set 6[1]

[1]Made by Einar Rønquist, Department of Mathematical Sciences, NTNU, N-7491 Trondheim, Norway.

Exercise 1

(January 5, 2004)

a)



$$H_1^k = 1 - x/h - y/h$$
$$H_2^k = x/h$$
$$H_3^k = y/h$$
$$\underline{H_\alpha^k = c_\alpha + c_{x_\alpha} \cdot x + c_{y_\alpha} \cdot y, \quad \alpha = 1, 2, 3}$$

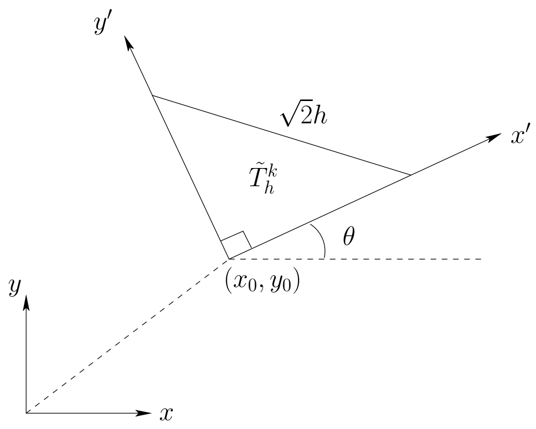
$$\begin{aligned}
H_1^k &= 1 - \frac{x}{h} - \frac{y}{h} &&\Rightarrow\quad c_1 = \frac{1}{h}, \quad c_{x_1} = -\frac{1}{h}, \quad c_{y_1} = -\frac{1}{h} \\
H_2^k &= \frac{x}{h} &&\Rightarrow\quad c_2 = 0, \quad c_{x_2} = \frac{1}{h}, \quad c_{y_2} = 0 \\
H_3^k &= \frac{y}{h} &&\Rightarrow\quad c_3 = 0, \quad c_{x_3} = 0, \quad c_{y_3} = \frac{1}{h}
\end{aligned}$$

$$\begin{aligned}
A_{\alpha\beta}^k &= \int_{T_h^k} \left(\frac{\partial H_\alpha^k}{\partial x}\frac{\partial H_\beta^k}{\partial x} + \frac{\partial H_\alpha^k}{\partial y}\frac{\partial H_\beta^k}{\partial y}\right) \mathrm{d}A \\
&= \mathrm{Area}^k (c_{x_\alpha} c_{x_\beta} + c_{y_\alpha} c_{y_\beta}) \qquad 1 \leq \alpha, \beta \leq 3 \\
A_{11}^k &= \frac{h^2}{2} \cdot ((-1)\cdot(-1) + (-1)\cdot(-1)) \frac{1}{h^2} = \frac{1}{2} \cdot 2 \\
A_{12}^k &= \frac{h^2}{2} \cdot ((-1)\cdot 1 + (-1)\cdot 0) \frac{1}{h^2} = \frac{1}{2} \cdot (-1) \\
A_{13}^k &= \frac{h^2}{2} \cdot ((-1)\cdot 0 + (-1)\cdot 1) \frac{1}{h^2} = \frac{1}{2} \cdot (-1) \\
A_{21}^k &= \frac{h^2}{2} \cdot (1\cdot(-1) + 0\cdot(-1)) \frac{1}{h^2} = \frac{1}{2} \cdot (-1) \\
A_{22}^k &= \frac{h^2}{2} \cdot (1\cdot 1 + 0\cdot 0) \frac{1}{h^2} = \frac{1}{2} \cdot (1) \\
A_{23}^k &= \frac{h^2}{2} \cdot (1\cdot 0 + 0\cdot 1) \frac{1}{h^2} = \frac{1}{2} \cdot (0) \\
A_{31}^k &= \frac{h^2}{2} \cdot (0\cdot(-1) + 1\cdot(-1)) \frac{1}{h^2} = \frac{1}{2} \cdot (-1) \\
A_{32}^k &= \frac{h^2}{2} \cdot (0\cdot 1 + 1\cdot 0) \frac{1}{h^2} = \frac{1}{2} \cdot (0) \\
A_{33}^k &= \frac{h^2}{2} \cdot (0\cdot 0 + 1\cdot 1) \frac{1}{h^2} = \frac{1}{2} \cdot (1)
\end{aligned}$$

As expected $A^k_{\alpha\beta} = A^k_{\beta\alpha}$ (symmetry).

$$\underline{\underline{A^k = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}}$$

b)



Consider a right-triangular element with hypotenuse $\sqrt{2}h$ at <u>any</u> position and <u>any</u> orientation.

First, notice that

$$\underline{A}^k_{\alpha\beta} = \int_{T_h^k} \nabla H^k_\alpha \cdot \nabla H^k_\beta \, \mathrm{d}A$$

The integrand $(\nabla H^k_\alpha \cdot \nabla H^k_\beta)$ represents a dot product. We know that this is a <u>geometric invariant</u> in the sense that the dot product is unaffected by the choice of Cartesian coordinate system. Hence, we can introduce another Cartesian coordinate system (x', y') which represents a <u>translation</u> and a <u>rotation</u> compared to (x, y).
In the (x', y') coordinate system, the local basis functions read:

$$\begin{aligned} H^k_1 &= 1 - \frac{x'}{h} - \frac{y'}{h} \\ H^k_2 &= \frac{x'}{h} \\ H^k_3 &= \frac{y'}{h} \end{aligned}$$

and

$$\underline{A}^k = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad \text{as in } \mathbf{a)}$$

c)

$$\begin{aligned} M^k_{\alpha\beta} &= \int_{T_h^k} \varphi_i|_{T_h^k}\ \varphi_j|_{T_h^k}\,\mathrm{d}A && \begin{array}{c}\text{for some } i,j.\\ \text{(overlapping basis functions)}\end{array} \\ &= \int_{T_h^k} H^k_\alpha \cdot H^k_\beta\,\mathrm{d}A, && 1 \le \alpha,\beta \le 3 \end{aligned}$$

Consider the element depicted in **a)**.

$$\begin{aligned} H^k_1 &= 1 - \frac{x}{h} - \frac{y}{h} \\ H^k_2 &= \frac{x}{h} \\ H^k_3 &= \frac{y}{h} \end{aligned}$$

$$\begin{aligned} M^k_{11} &= \int_{T_h^k} (1 - \frac{x}{h} - \frac{y}{h})^2\,\mathrm{d}A = h^2 \int_{\xi=0}^{1}\int_{\eta=0}^{1-\xi} (1-\xi-\eta)^2\,\mathrm{d}\eta\,\mathrm{d}\xi \\ &= h^2 \int_{\xi=0}^{1} [-\frac{1}{3}(1-\xi-\eta)^3]_0^{1-\xi}\,\mathrm{d}\xi \\ &= h^2 \int_{\xi=0}^{1} [-\frac{1}{3}(1-\xi-(1-\xi))^3 + \frac{1}{3}(1-\xi)^3]\,\mathrm{d}\xi \\ &= h^2 \int_{\xi=0}^{1} \frac{1}{3}(1-\xi)^3\,\mathrm{d}\xi = h^2\frac{1}{3}[-\frac{1}{4}(1-\xi)^4]_0^1 \\ &= h^2 \cdot \frac{1}{3}\cdot\frac{1}{4} \\ &= \underline{\frac{1}{12}h^2} \end{aligned}$$

$$\begin{aligned} M^k_{12} &= \int_{T_h^k} (1 - \frac{x}{h} - \frac{y}{h})\frac{x}{h}\,\mathrm{d}x\,\mathrm{d}y = h^2 \int_{\xi=0}^{1}\int_{\eta=0}^{1-\xi} (1-\xi-\eta)\xi\,\mathrm{d}\eta\,\mathrm{d}\xi \\ &= h^2 \int_{\xi=0}^{1} \xi[-\frac{1}{2}(1-\xi-\eta)^2]_0^{1-\xi}\,\mathrm{d}\xi \\ &= h^2 \int_{\xi=0}^{1} \xi\frac{1}{2}(1-\xi)^2\,\mathrm{d}\xi = \frac{h^2}{2}\int_0^1 (\xi - 2\xi^2 + \xi^3)\,\mathrm{d}\xi \\ &= \frac{h^2}{2}[\frac{1}{2}\xi^2 - \frac{2}{3}\xi^3 + \frac{1}{4}\xi^4]_0^1 = \frac{h^2}{2}(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}) = \frac{h^2}{2}\cdot\frac{1}{12} \\ &= \underline{\frac{h^2}{24}} \end{aligned}$$

$$
\begin{aligned}
M_{13}^k &= \int_{T_h^k} (1 - \frac{x}{h} - \frac{y}{h}) \frac{y}{h} \,\mathrm{d}x \,\mathrm{d}y = h^2 \int_{\xi=0}^{1} \int_{\eta=0}^{1-\xi} (1 - \xi - \eta)\eta \,\mathrm{d}\eta \,\mathrm{d}\xi \\
&= h^2 \int_{\xi=0}^{1} [\frac{1}{2}(1-\xi)\eta^2 - \frac{1}{3}\eta^3]_0^{1-\xi} \,\mathrm{d}\xi \\
&= h^2 \int_{\xi=0}^{1} (\frac{1}{2}(1-\xi)^3 - \frac{1}{3}(1-\xi)^3) \,\mathrm{d}\xi = h^2 \int_0^1 \frac{1}{6}(1-\xi)^3 \,\mathrm{d}\xi \\
&= \frac{h^2}{6}[-\frac{1}{4}(1-\xi)^4]_0^1 = \frac{h^2}{6} \cdot \frac{1}{4} \\
&= \underline{\frac{h^2}{24}}
\end{aligned}
$$

$$
\begin{aligned}
M_{22} &= \int_{T_h^k} (\frac{x}{h})^2 \,\mathrm{d}x \,\mathrm{d}y = h^2 \int_{\xi=0}^{1} \int_{\eta=0}^{1-\xi} \xi^2 \,\mathrm{d}\eta \,\mathrm{d}\xi \\
&= h^2 \int_0^1 \xi^2 (1-\xi) \,\mathrm{d}\xi = h^2 [\frac{1}{3}\xi^3 - \frac{1}{4}\xi^4]_0^1 \\
&= \underline{\frac{h^2}{12}}
\end{aligned}
$$

$$
\begin{aligned}
M_{23}^k &= \int_{T_h^k} (\frac{x}{h})(\frac{y}{h}) \,\mathrm{d}x \,\mathrm{d}y = h^2 \int_{\xi=0}^{1} \int_{\eta=0}^{1-\xi} \xi \cdot \eta \,\mathrm{d}\eta \,\mathrm{d}\xi \\
&= h^2 \int_{\xi=0}^{1} \xi [\frac{1}{2}\eta^2]_0^{1-\xi} \,\mathrm{d}\xi = h^2 \int_0^1 \xi \frac{1}{2}(1-\xi)^2 \,\mathrm{d}\xi \\
&= \frac{h^2}{2} \int_0^1 (\xi - 2\xi^2 + \xi^3) \,\mathrm{d}\xi = \frac{h^2}{2} [\frac{1}{2}\xi^2 - \frac{2}{3}\xi^3 + \frac{1}{4}\xi^4]_0^1 \\
&= \underline{\frac{h^2}{24}}
\end{aligned}
$$

$$
M_{33}^k = \underline{\frac{h^2}{12}}
$$

$$
\underline{\underline{M}^k = \frac{h^2}{24} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}}
$$

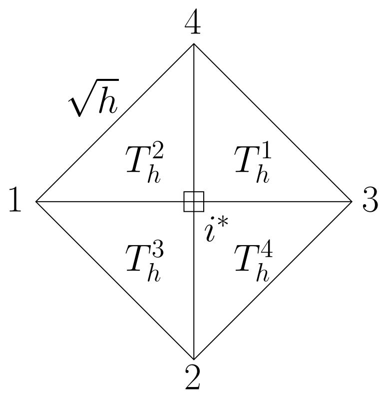
(Invariant with respect to translation and rotation.)

Check:

$$
\begin{aligned}
\underline{1}^T \underline{M}^k \underline{1} &= \sum_\alpha \sum_\beta M^k_{\alpha\beta} \\
&= \frac{h^2}{2} \\
&= \underline{\text{Area}^k}
\end{aligned}
$$

Exercise 2

(January 5, 2004)



T_h^1 : global nodes $(i^*, 3, 4)$
T_h^2 : global nodes $(i^*, 4, 1)$
T_h^3 : global nodes $(i^*, 1, 2)$
T_h^4 : global nodes $(i^*, 2, 3)$

Initial row i^* in $\underline{\tilde{A}}_h$:

Column:	1	2	3	4	i^*
row i^* $\rightarrow$	0	0	0	0	0

Row i^* after assembling contributions from T_h^1:

row i^* $\rightarrow$			$-\frac{1}{2}$	$-\frac{1}{2}$	1

Row i^* after assembling contributions from T_h^2:

row i^* $\rightarrow$	$-\frac{1}{2}$		$-\frac{1}{2}$	-1	2

Row i^* after assembling contributions from T_h^3:

row i^* $\rightarrow$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	3

Row i^* after assembling contributions from T_h^4 (final result):

row i^* $\rightarrow$	-1	-1	-1	-1	4