TMA4220
Numerical solution of partial differential equations
by element methods

Suggested solutions to Problem Set 8[1]

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Exercise 1

We first express the element matrix associated with element T_h^k as a sum of two contributions: one from the diffusion term and one from the convection term,

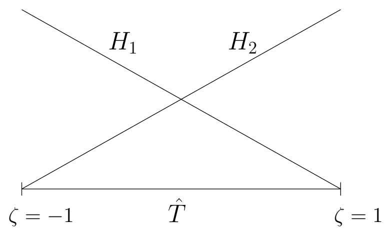
$$\underline{A}_h^k = \underbrace{\underline{A}_{\text{Laplace}}^k}_{\text{diffusion}} + \underbrace{\underline{C}^k}_{\text{convection}}$$

We have already seen how to derive

$$\underline{A}_{\text{Laplace}}^k = \frac{\kappa}{h^k}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

On page 5 in the notes dated March 9, 2004, we derived the result

$$C_{\alpha\beta}^k = \int_{\hat{T}} U \frac{\mathrm{d}H_\beta}{\mathrm{d}\zeta} H_\alpha \,\mathrm{d}\zeta \qquad \begin{array}{l}\alpha = 1,2 \\ \beta = 1,2\end{array}$$



$$H_1(\zeta) = \frac{1-\zeta}{2}, \quad \frac{\mathrm{d}H_1}{\mathrm{d}\zeta} = -\frac{1}{2}$$
$$H_2(\zeta) = \frac{1+\zeta}{2}, \quad \frac{\mathrm{d}H_2}{\mathrm{d}\zeta} = +\frac{1}{2}$$

Explicit computation of $C_{\alpha\beta}^k$, $\alpha, \beta = 1,2$ then gives:

$$C_{11}^k = \int_{-1}^{1} U \frac{\mathrm{d}H_1}{\mathrm{d}\zeta} H_1 \,\mathrm{d}\zeta \qquad (\text{U} = \underline{\text{constant}} \text{ in } \mathbb{R}^1\text{; a given incompressible velocity field })$$

$$= U \int_{-1}^{1} (-\frac{1}{2})(\frac{1-\zeta}{2}) \,\mathrm{d}\zeta = \frac{U}{4}\int_{-1}^{1} (\zeta - 1)\,\mathrm{d}\zeta = \frac{U}{4}\cdot(-2)$$

$$\underline{C_{11}^k = -\frac{U}{2}}$$

$$C_{12}^k = U \int_{-1}^{1} \frac{\mathrm{d}H_2}{\mathrm{d}\zeta} H_1 \,\mathrm{d}\zeta = U\int_{-1}^{1} (\frac{1}{2})(\frac{1-\zeta}{2})\,\mathrm{d}\zeta = \frac{U}{4}\cdot(+2)$$

$$\underline{C_{12}^k = +\frac{U}{2}}$$

$$\underline{C_{21}^k} = U \int_{-1}^{1} \frac{\mathrm{d}H_1}{\mathrm{d}\zeta} H_2 \,\mathrm{d}\zeta = U\int_{-1}^{1} (-\frac{1}{2})(\frac{1+\zeta}{2})\,\mathrm{d}\zeta = \underline{-\frac{U}{2}}$$

$$\underline{C_{22}^k} = U \int_{-1}^{1} \frac{\mathrm{d}H_2}{\mathrm{d}\zeta} H_2 \,\mathrm{d}\zeta = U\int_{-1}^{1} (+\frac{1}{2})(\frac{1+\zeta}{2})\,\mathrm{d}\zeta = \underline{+\frac{U}{2}}$$

Hence,

$$\underline{C}^k = \frac{U}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

and

$$\underline{A}_h^k = \frac{\kappa}{h^k}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{U}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

Exercise 2

$$\begin{aligned} -\kappa u_{xx} - U u_x &= 0 \qquad \text{in } \Omega = (0,\infty) \\ u(0) &= 0 \\ u(\infty) &= 0 \end{aligned}$$

Second-order central differences gives (see the notes dated March 9, 2004):

$$-\varepsilon \frac{(\hat{u}_{j+1} - 2\hat{u}_j + \hat{u}_{j-1})}{\Delta x^2} - \frac{(\hat{u}_{j+1} - \hat{u}_{j-1})}{2\Delta x} = 0 \ .$$

As derived in class, the exact difference solution can be expressed as:

$$\underline{\hat{u}_j = (\frac{1 - P_g/2}{1 + P_g/2})^j, \qquad j = 0, 1, \ldots}$$

Here, the grid Peclet number $P_g = \frac{\Delta x}{\varepsilon} = \frac{U \Delta x}{\kappa} > 0$.

Oscillations (defined as $\hat{u}_j$ being both negative and positive) will occur when

$$1 - P_g/2 < 0 \Rightarrow \underline{P_g > 2}$$

This corresponds to $\underline{\Delta x > 2\varepsilon}$, i.e., we do not resolve the true boundary layer of thickness ε.

Exercise 3

Modified problem:

$$-(\varepsilon + \frac{\Delta x}{2}) u_{xx} - u_x = 0$$

Second-order finite differences applied to the modified problem gives:

$$\begin{aligned} -(\varepsilon + \frac{\Delta x}{2})\frac{(\hat{u}_{j+1} - 2\hat{u}_j + \hat{u}_{j-1})}{\Delta x^2} - \frac{(\hat{u}_{j+1} - \hat{u}_{j-1})}{2\Delta x} &= 0 \\ -\varepsilon \frac{(\hat{u}_{j+1} - 2\hat{u}_j + \hat{u}_{j-1})}{\Delta x^2} - \underbrace{\frac{(\hat{u}_{j+1} - 2\hat{u}_j + \hat{u}_{j-1})}{2\Delta x} - \frac{(\hat{u}_{j+1} - \hat{u}_{j-1})}{2\Delta x}}_{= \frac{\hat{u}_{j+1} - \hat{u}_j}{\Delta x}} &= 0 \end{aligned}$$

i.e., first order upwinding applied to the original problem.

The grid Peclet number for the modified problem ($\Leftrightarrow$ first order upwinding) is:

$$\tilde{P}_g = \frac{\Delta x}{\tilde{\varepsilon}} = \frac{\Delta x}{\varepsilon + \frac{\Delta x}{2}}$$

$\max\limits_{\varepsilon} \tilde{P}_g = 2 \quad \Rightarrow$ first order upwinding is always stable (no oscillations), but we will experience a loss of accuracy. As ε goes to zero ($\varepsilon \ll \Delta x$), the numerical solution will have a boundary layer of thickness approximately equal to $\Delta x/2$, i.e., the error in the boundary layer region will be $\mathcal{O}(1)$. However, the outer solution will be good.