TMA 4220

The steady convection-diffusion equation

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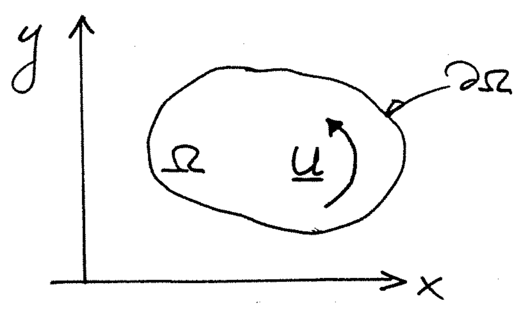


General problem:

$\underline{U}$: given velocity field

$\nabla\cdot\underline{U} = 0$ (incompressible fluid — mass balance)

u = Temperature

Solve the steady convection-diffusion equation (energy balance):

$$\begin{cases} \underline{U}\cdot\nabla u = \varkappa\nabla^2 u + f & \text{in } \Omega \\ u \text{ prescribed on } \partial\Omega & \end{cases}$$

<u>Note</u>: $\underline{U} = \underline{0} \Rightarrow -\varkappa\nabla^2 u = f$ in Ω

→ steady heat equation (Poisson)

<u>$\mathbb{R}^1$</u>: $\nabla\cdot\underline{U} = 0 \Rightarrow \underline{U} = U =$ constant

<u>Model problem</u>:

$$\begin{cases} -\varkappa u_{xx} + U\cdot u_x = f & \text{in } \Omega = (0,1) \\ u(0) = u(1) = 0 & \end{cases}$$

<u>No</u> equivalent minimization statement.

However, we can still use the Galerkin statement or weak formulation which can be derived as follows:

Find $u \in X$ such that

$$\int_0^1 (-\varkappa u_{xx} + U u_x - f)\, v \, dx = 0 \qquad \forall v \in X$$

or

$$\left[\int_0^1 (\mathcal{L}u - f)\, v \, dx = 0 \qquad \forall v \in X \right]$$

Integrate the diffusion term by parts:

$$\left[-\varkappa u_x v\right]_0^1 + \int_0^1 (\varkappa u_x v_x + U u_x v)\, dx = \int_0^1 f v \, dx \qquad \forall v \in X$$

$= 0$

$(X = H_0^1(\Omega))$

Find $u \in X = H_0^1(\Omega)$ such that

$$\int_0^1 (\varkappa u_x v_x + U u_x v)\, dx = \int_0^1 f v \, dx \qquad \forall v \in X$$

$\underline{U = 0}$ $\Rightarrow$ The problem reduces to:

Find $u \in X = H_0^1(\Omega)$ such that

$$a(u, v) = \ell(v) \qquad \forall v \in X$$

where

$$a(w, v) = \int_0^1 \varkappa u_x v_x \, dx \qquad \underline{\text{SPD}}$$

$$\ell(v) = \int_0^1 f v \, dx \qquad \underline{\text{bounded}}$$

<u>$U \neq 0$</u>: We can still write the problem in the form:

$$\begin{cases} \text{Find } u \in X = H_0^1(\Omega) \text{ such that} \\ a(u,v) = l(v) \qquad \forall v \in X \end{cases}$$

However, the bilinear form is now

$$a(w,v) = \int_0^1 (\varkappa\, w_x v_x + U \cdot w_x v)\, dx$$

<u>Observation</u>: Loss of symmetry

$$a(w,v) \neq a(v,w)$$

We have still <u>gained</u> compared to the strong formulation:

- Lowering of the regularity requirement on the solution u ($u \in H^1$ as opposed to $u \in H^2$)
- The weak formulation is more general in terms of admissible data f
- Natural imposition of Neumann boundary conditions.

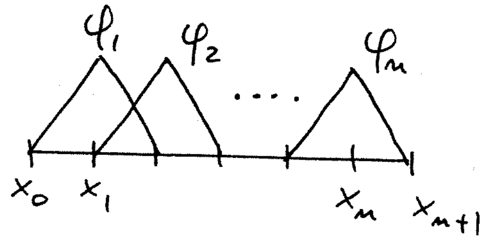
Approximation (finite-dimensional problem)

Find $u_h \in X_h \subset X$ such that

$$a(u_h, v) = \ell(v) \qquad \forall v \in X_h$$

$$X_h = \{ v \in H^1(\Omega) \,/\, v(0) = v(1) = 0,\ v|_{T_h^k} \in \mathbb{P}_1(T_h^k),\ k = 1, \dots, K \}$$

$$= \operatorname{span} \{ \varphi_1, \varphi_2, \dots, \varphi_n \}$$



Discrete equations

$$\underline{A}_h \underline{u}_h = \underline{F}_h$$

$$(A_h)_{ij} = \int_0^1 \left(\varepsilon \frac{d\varphi_j}{dx} \frac{d\varphi_i}{dx} + U \frac{d\varphi_j}{dx} \varphi_i \right) dx = a(\varphi_j, \varphi_i)$$

$$(F_h)_i = \int_0^1 f \varphi_i \, dx = \ell(\varphi_i)$$

Here:
$$u_h(x) = \sum_{j=1}^{n} u_{hj} \varphi_j(x)$$

$$\underline{u}_h = \begin{bmatrix} u_{h1} \\ u_{h2} \\ \vdots \\ u_{hn} \end{bmatrix}$$

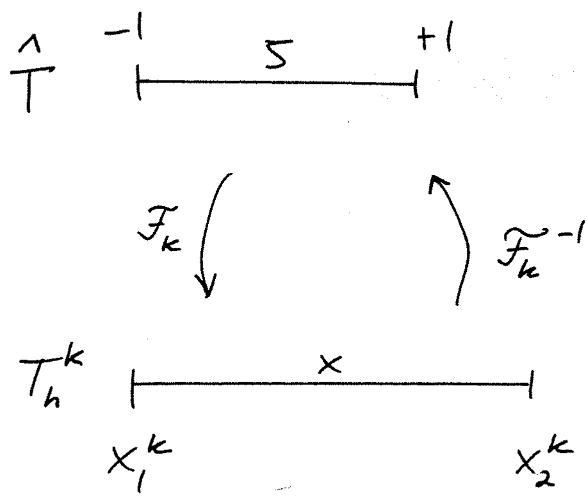
Consider

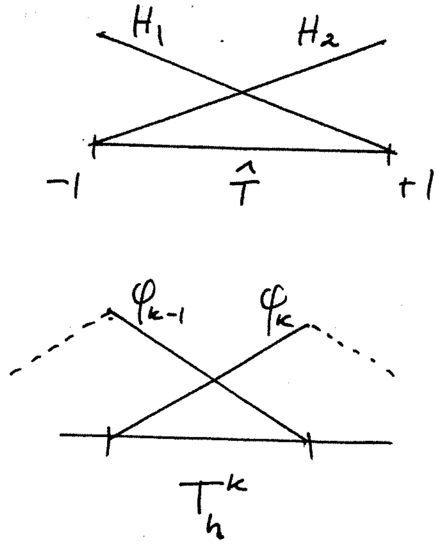
$$\int_0^1 u \,\frac{d\varphi_j}{dx}\,\varphi_i \, dx = \sum_{k=1}^{K} \int_{T_h^k} u \,\frac{d\varphi_j}{dx}\,\varphi_i \, dx \qquad 1 \le i,j \le n$$

$$= \sum_k \int_{T_h^k} u \,\frac{dH_\beta}{dx}\left(\mathcal{F}_k^{-1}(x)\right) \cdot H_\alpha\left(\mathcal{F}_k^{-1}(x)\right) dx$$

$$= \sum_k \int_{\hat{T}} u \,\frac{dH_\beta}{d\xi}(\xi) \cdot \frac{d\xi}{dx} \cdot H_\alpha(\xi)\,\frac{dx}{d\xi}\, d\xi$$

$$= \sum_k \int_{\hat{T}} u \,\frac{dH_\beta}{d\xi}\, H_\alpha \, d\xi \qquad , \ \alpha,\beta = 1,2$$





Affine mapping

$\xi = \mathcal{F}_k^{-1}(x)$

$x = \mathcal{F}_k(\xi)$

$U = 0$:
$$A^k_{\alpha\beta} = \frac{\varkappa}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$U \neq 0$:
$$A^k_{\alpha\beta} = \frac{\varkappa}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{U}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

<u>Exercise</u>: Prove the last result ($U \neq 0$)

Assemble elemental contributions:

$$\begin{bmatrix} 1 & -1 & \\ -1 & 1 \; +1 & -1 \\ & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & \\ -1 & 1 \; -1 & 1 \\ & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$$

Consider $-\varkappa u_{xx} + U u_x = 0$ in $\Omega = (0,1)$

A typical row in the finite element formulation will thus read:

$$\frac{\varkappa}{h}\left(-u_{i-1} + 2u_i - u_{i+1}\right) + \frac{U}{2}\left(u_{i+1} - u_{i-1}\right) = 0$$

or

$$-\varkappa \frac{\left(u_{i-1} - 2u_i + u_{i+1}\right)}{h^2} + U \frac{\left(u_{i+1} - u_{i-1}\right)}{2h} = 0$$

This is identical to a second-order finite difference scheme. (uniform mesh).

We will now continue to analyze this one-dimensional model problem in the context of finite differences. We will return to the finite element case later.

The steady convection-diffusion equation can be written as

given data

$$\underbrace{-\varkappa \nabla^2 u}_{} + \underbrace{\underline{U} \cdot \nabla u}_{} = f \quad \text{in } \Omega.$$

diffusion term

represents the highest order differential operator (second-order), which dictates the type of PDE (elliptic) and the type of boundary conditions we have to specify.

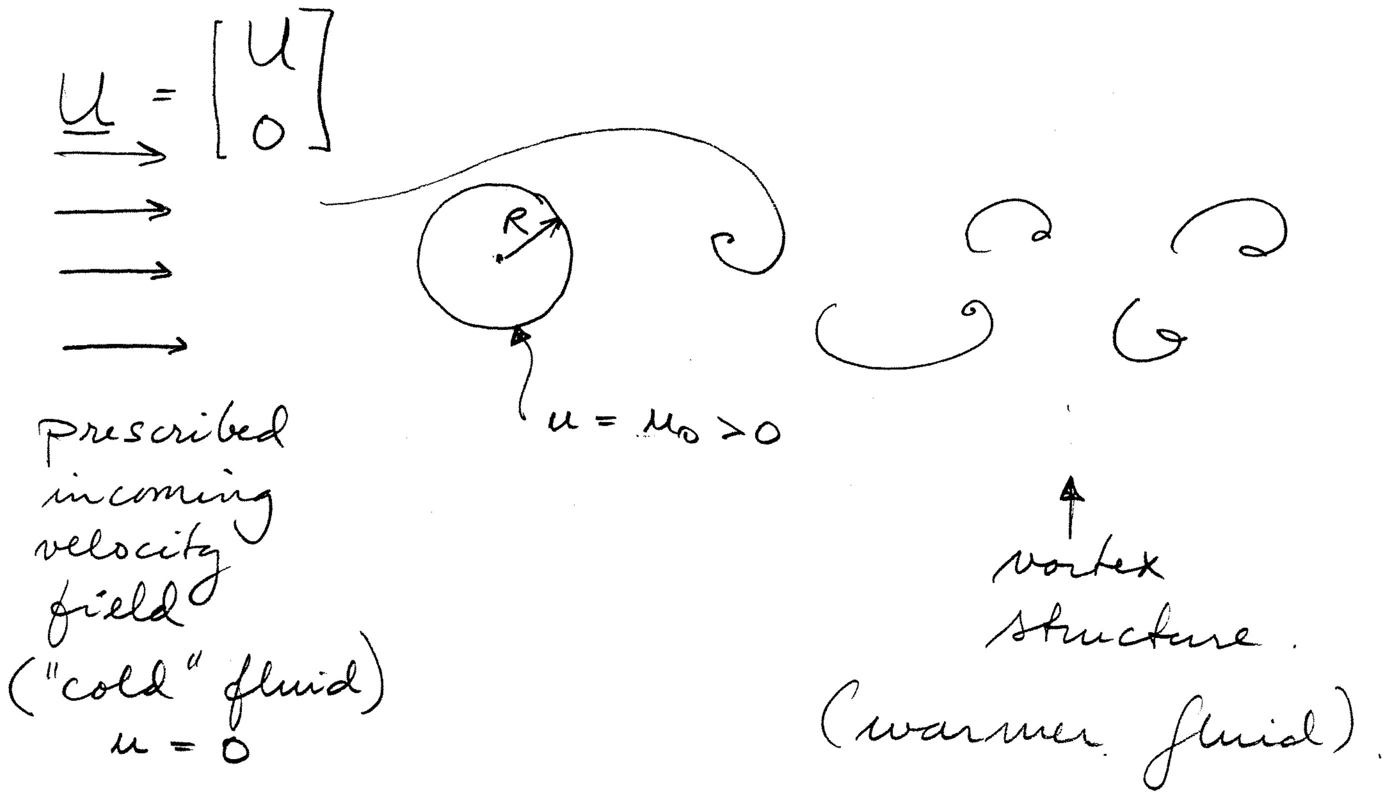
convection term

$\underline{U}$ is here a prescribed velocity field (a vector).

For an incompressible fluid, $\nabla \cdot \underline{U} = 0$.

Non-dimensionalization:

Example: Flow past a heated cylinder with radius



$$|\varkappa \nabla^2 u| \sim \frac{\varkappa u_0}{R^2}$$

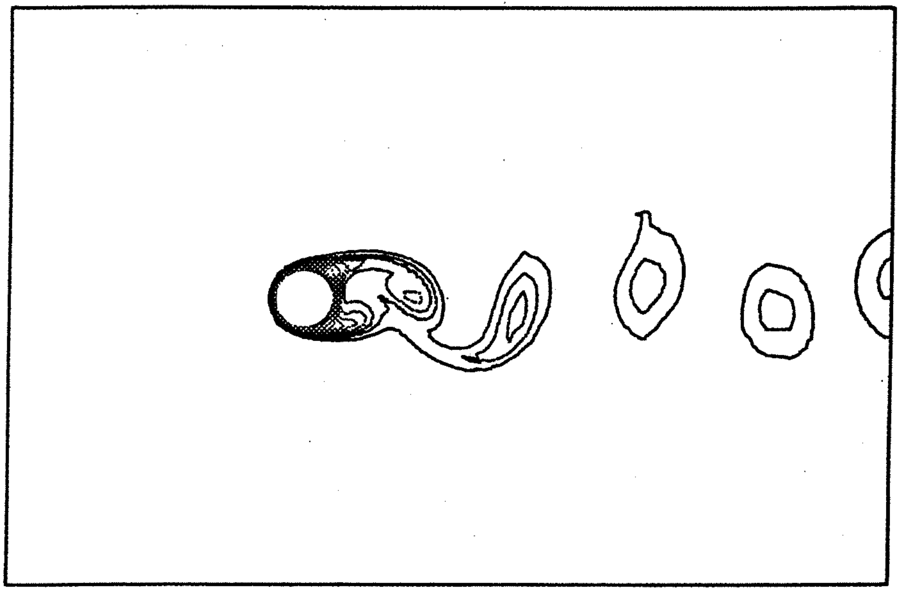
$$|\underline{U} \cdot \nabla u| \sim \frac{U u_0}{R}$$

$$\frac{|\text{conv. term}|}{|\text{diff. term}|} \sim \frac{\frac{U \cdot u_0}{R}}{\frac{\varkappa u_0}{R^2}} = \frac{U \cdot R}{\varkappa} = Pe$$

Pe : Peclet number (non-dimensional)

Computational results (an example)

- unsteady convection diffusion
- $Pe = 200$ ($Re = 200$)



The figure shows the computational domain and temperature contours at a particular intant in time.
(The vortex structure is steady periodic.)

In general:

$$\boxed{Pe = \frac{U \cdot L}{\varkappa}}$$

U : a velocity scale
L : a length scale
$\varkappa$: diffusivity

<u>Interpretation</u>:

$Pe \ll 1$: diffusion dominates
$Pe \gg 1$: convection dominates.

A one-dimensional case:

length of the domain

$$(*)\quad \begin{cases} -\varkappa u_{xx} + U u_x = f & \text{in } \Omega = (0, L_D) \\ u(0) = 1, \quad u(L_D) = 0. \end{cases}$$

Note that, in $\mathbb{R}^1$, the prescribed velocity field $\underline{U}$ must be a constant (U) if we insist that the velocity field be incompressible:

$$\nabla \cdot \underline{U} = 0 \Rightarrow \frac{\partial U_x}{\partial x} = 0 \Rightarrow \underline{U_x = U = \text{constant}}$$

We can write (*) as

$$\boxed{-\varepsilon u_{xx} + u_x = f/U \quad \text{in } \Omega = (0, L_D)}$$

with

$$\varepsilon = \frac{\varkappa}{U}, \quad [\varepsilon] = \text{length}$$

If

$$\varepsilon << L < L_D$$

length scale ↑ ↑ length of the domain

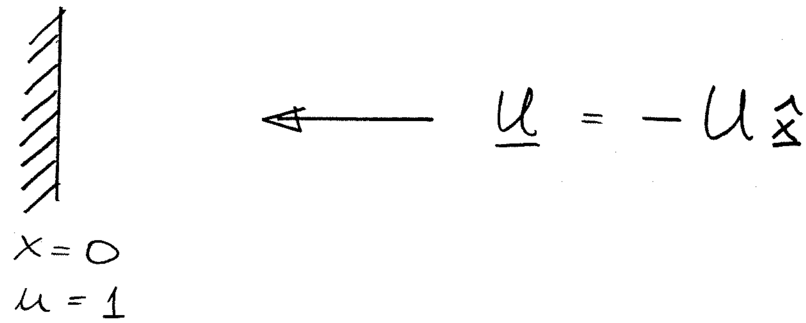
$$\Rightarrow \frac{L}{\varepsilon} >> 1 \quad \text{or} \quad \frac{UL}{\varkappa} = Pe >> 1$$

Model problem ($\mathbb{R}^1$)

$$-\varepsilon u_{xx} - u_x = 0 \quad \text{in } \Omega = (0, \infty)$$

$$u(0) = 1, \quad u(\infty) = 0$$

perforated wall

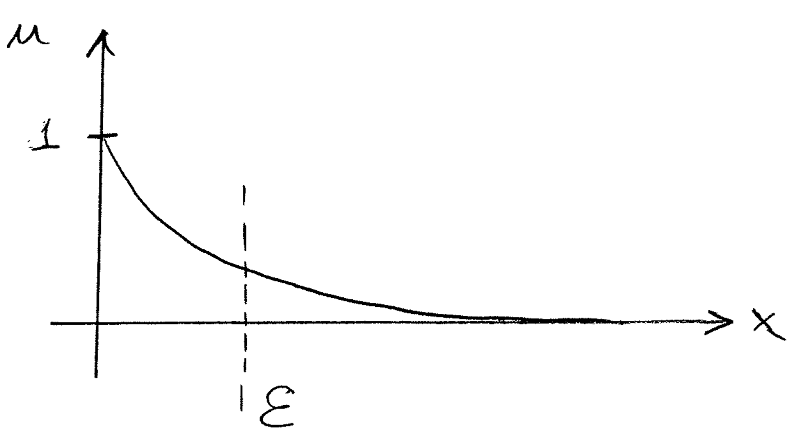


Recall that $\varepsilon = \dfrac{\varkappa}{U}$

Note that $f = 0$ in this case.

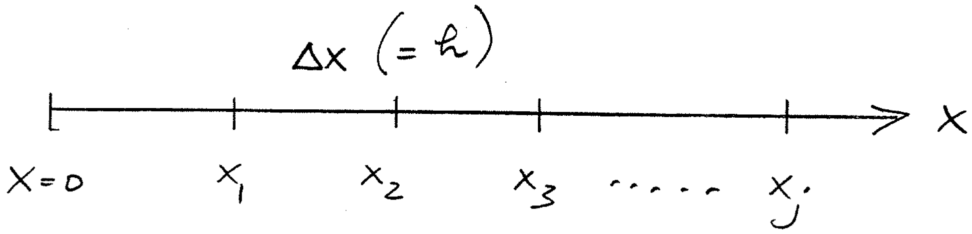
Exact solution

$$u(x) = e^{-x/\varepsilon}$$



boundary layer of thickness $O(\varepsilon)$

Discretization



$\downarrow$

FDM (second order) $\Longleftrightarrow$ FEM (linear elements)

(uniform mesh)

$\downarrow$

$$-\varepsilon \frac{(\hat{u}_{j+1} - 2\hat{u}_j + \hat{u}_{j-1})}{\Delta x^2} - \frac{(\hat{u}_{j+1} - \hat{u}_{j-1})}{2\Delta x} = 0$$

$$j = 1, 2, \ldots$$

$\hat{u}_0 = 1$

$\hat{u}_j \to 0$ as $j \to \infty$

Here,

$\hat{u}_j$ is the approximate FD (finite difference) solution at $x_j = j\Delta x$

Define the *grid* Peclet number as

$$P_g = \frac{\Delta x}{\varepsilon} = \frac{U \Delta x}{\varkappa}$$

(compare with the usual Peclet number
$$Pe = \frac{UL}{\varkappa}$$)

<u>Note</u>: P_g gives the ratio of the mesh size Δx relative to the boundary layer thickness ε.

The difference equations can now be expressed as

$$\left(1 + \frac{P_g}{2}\right)\hat{u}_{j+1} - 2\hat{u}_j + \left(1 - \frac{P_g}{2}\right)\hat{u}_{j-1} = 0$$

$$\hat{u}_0 = 1$$
$$\hat{u}_j \to 0 \ , \quad j \to \infty$$

which has the exact solution

$$\hat{u}_j = \left(\frac{1 - P_g/2}{1 + P_g/2}\right)^j \ , \quad j = 0, 1, \ldots$$

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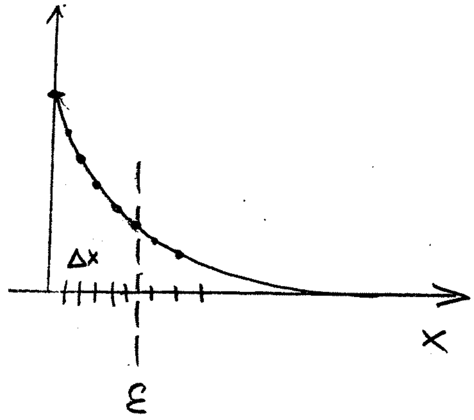
exact FD solution

Consider the cases

$\underline{P_g << 1}$

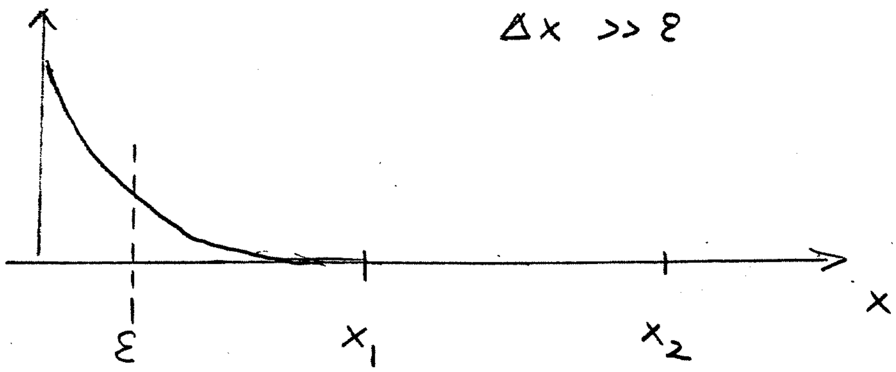


$\Delta x << \varepsilon$

$|e_j| \sim O(\Delta x^2)$

$(e_j = u_j - \hat{u}_j)$
error

$\underline{P_g >> 1}$



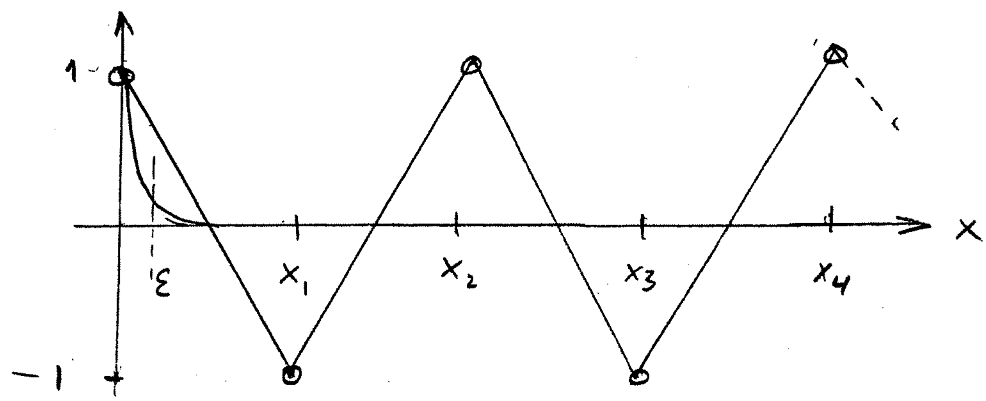
$\Delta x >> \varepsilon$

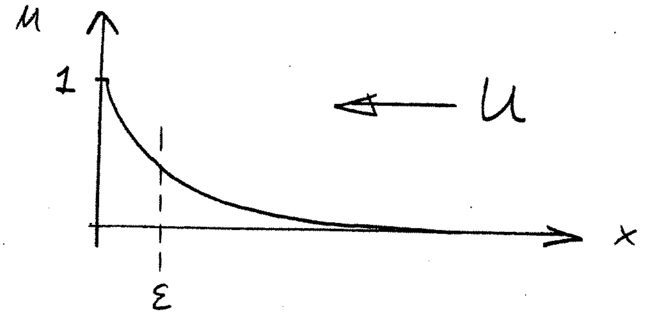
unresolved boundary layer

$P_g \to \infty$

$\hat{u}_j \to (-1)^j$

$|\hat{e}_j| \sim O(1)$





$u(0)=1$

$u(\infty)=0$

$$-\varepsilon u_{xx} - u_x = 0 \quad , \quad \varepsilon = \frac{\varkappa}{U} \quad , \quad u(x) = e^{-x/\varepsilon}$$

Consider the <u>heat flux</u> q at $x=0$:

$$q = -\varkappa \nabla u = -\varkappa u_x = -\varepsilon U u_x = U u$$

$$q(0) = U u(0) = U \neq 0$$

The larger U is, the larger the heat transfer at the boundary $x=0$.

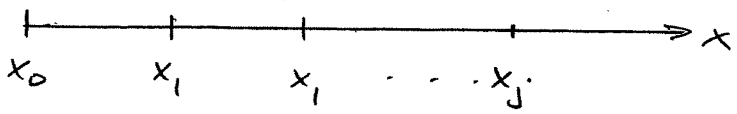
If $P_g >> 1$, $q(0) = \frac{2}{\Delta x}$ independent of U!

(non-physical results)

First-order upwinding

Again, consider the model problem

$$-\varepsilon u_{xx} - u_x = 0 \quad , \quad u(0) = 1, \quad u(\infty) = 0$$



$$-\varepsilon \underbrace{\left(\frac{\hat{u}_{j+1} - 2\hat{u}_j + \hat{u}_{j-1}}{\Delta x^2} \right)}_{\text{second-order difference scheme}} - \underbrace{\left(\frac{\hat{u}_{j+1} - \hat{u}_j}{\Delta x} \right)}_{\text{first order upwinding}} = 0 \quad , \quad j = 1, \ldots$$

As before: $\quad P_g = \dfrac{\Delta x}{\varepsilon} = \dfrac{U \Delta x}{\kappa}$

$$\Longrightarrow \quad (1 + P_g)\, \hat{u}_{j+1} - (2 + P_g)\hat{u}_j + \hat{u}_{j-1} = 0 \quad , \quad j = 1, \ldots$$

$$\hat{u}_0 = 1$$

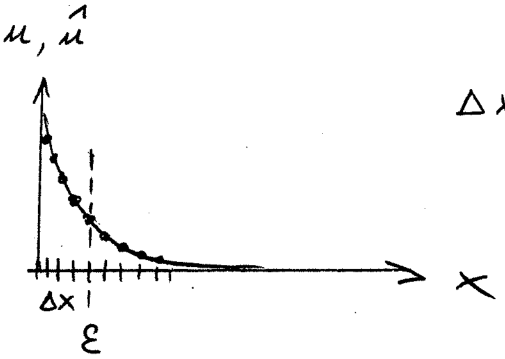
$$\hat{u}_j \to 0 \quad , \quad j \to \infty$$

$\Longrightarrow$ <u>exact</u> FD solution

$$\hat{u}_j = \left(\frac{1}{1 + P_g} \right)^j \quad , \quad j = 0, 1, \ldots$$

Again, consider the limiting cases.

$\underline{\underline{P_g \ll 1}}$



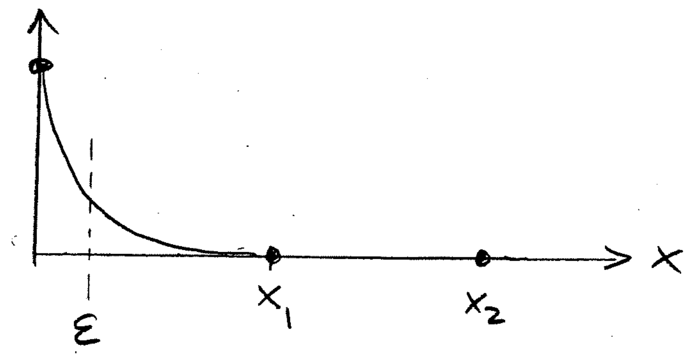
$\Delta x \ll \varepsilon$.

$$\left.\begin{array}{l}\text{For a } \underline{\text{fixed}}\ x = x_j \\ \text{and a } \underline{\text{fixed}}\ \varepsilon\end{array}\right\} \quad |e_j| \sim O(\Delta x)$$

first order

$\underline{P_g >> 1}$

$$P_g \to \infty \ , \quad \hat{u}_j = \left(\frac{1}{1+P_g}\right)^j \to \begin{cases} 1, & j = 0 \\ 0, & j > 0 \end{cases}$$



No oscillations

However, the boundary layer is $\underline{\text{not}}$ resolved

$|e_j| \sim O(1)$ near $x = 0$.

Consider the <u>modified problem</u>

$$-\left(\varepsilon + \frac{\Delta x}{2}\right) u_{xx} - u_x = 0 \quad \text{in } \Omega = (0, \infty)$$

"artificial" diffusion

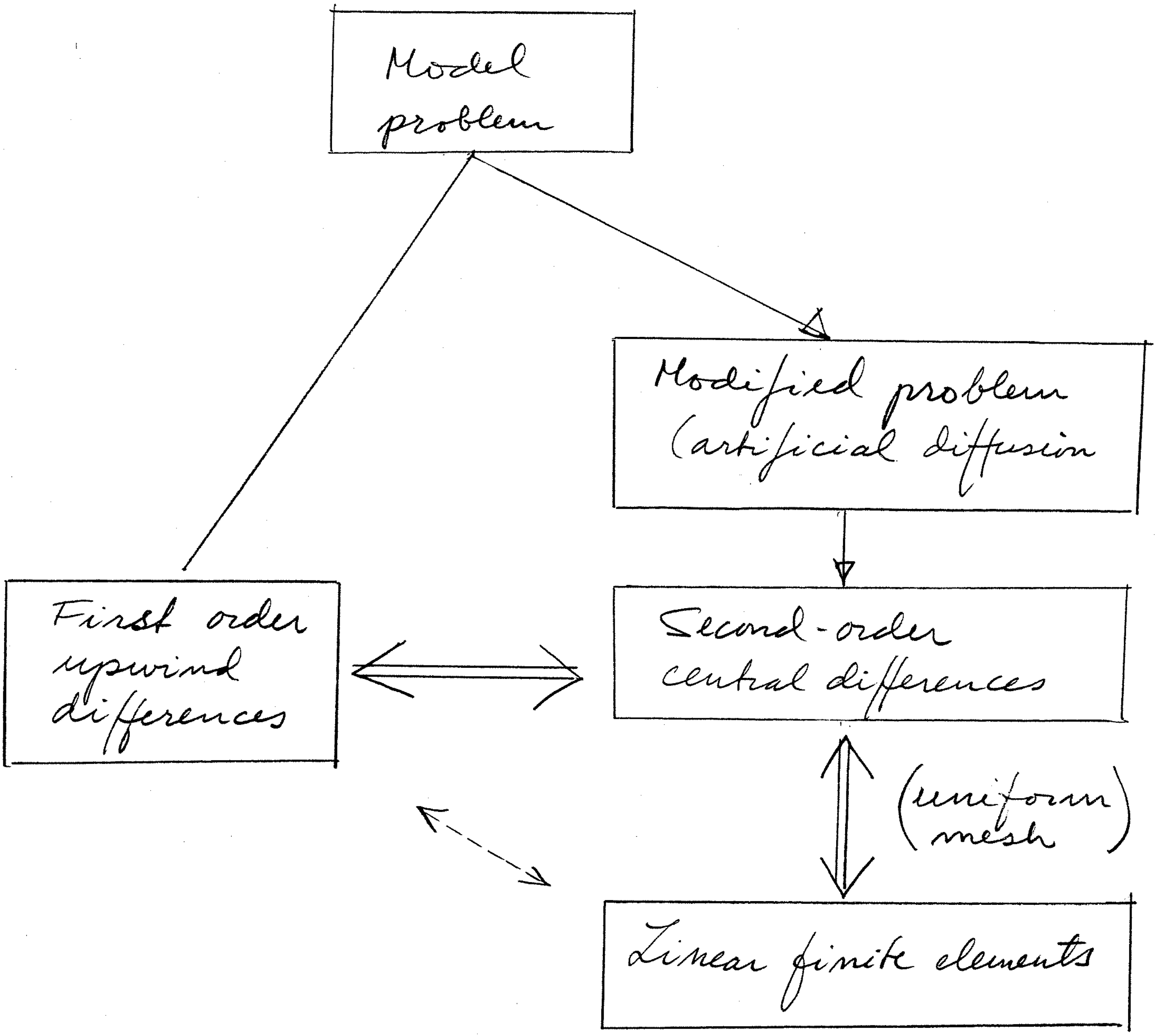
apply <u>centered</u> differences (second order)

$\Rightarrow$ obtain first-order upwinding scheme.
(the student should derive this).

This is the concept behind (streamline) upwinding for finite element methods:

1) Add diffusivity in the streamline direction only

2) Apply standard discretization procedure to the <u>modified</u> problem



Hence, in the finite element context, we can include the effect of upwinding by disretizing a modified problem where we have added a proper amount of artificial diffusion.

→ $\underline{\mathbb{R}^2, \mathbb{R}^3}$: Only add artificial diffusion in the <u>streamwise</u> direction