TMA4220: Numerical solution of partial differential equations by element methods

Steady Convection-Diffusion Equation: Finite Element Solution; Theory

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1 Model problem

Let us consider the one-dimensional convection-diffusion problem

$$\begin{aligned} -\kappa\, u_{xx} + U\, u_x &= f \qquad in\ \ \Omega = (0,1)\ , & (1)\\ u(0) &= 0\ , & (2)\\ u(1) &= 0\ . & (3) \end{aligned}$$

Here, u represents temperature, κ represents the thermal conductivity, f represents a volumetric heat source, and U is a constant. We can think of U as representing a divergence-free (or incompressible) velocity field which is given. Homogeneous Dirichlet boundary conditions are prescribed at $x = 0$ and $x = 1$. The length of the domain is $L = 1$.

The (dimensionless) Peclet number associated with the physical problem is

$$P = \frac{U\, L}{\kappa}\ , \qquad (4)$$

and it measures the importance of convection relative to diffusion.

1.1 Weak formulation

Let us now give the weak formulation of the one-dimensional convection-diffusion problem (1)-(3). To this end, we first define the usual function space

$$X = \{v \in H^1(\Omega) \mid v(0) = 0;\ v(1) = 0\} \equiv H_0^1(\Omega). \qquad (5)$$

The weak form of (1)-(3) can then be expressed as: Find $u \in X$ such that

$$a(u,v) = l(v) \qquad \forall v \in X\ , \qquad (6)$$

with

$$\begin{aligned} a(w,v) &= \int_0^1 (\kappa\, w_x v_x\ +\ U\, w_x v)\, dx & (7)\\ l(v) &= \int_0^1 f\, v\, dx\ . & (8) \end{aligned}$$

When $U = 0$, we recover the Poisson problem. Note that the bilinear form $a(\cdot,\cdot)$ is nonsymmetric when $U \neq 0$, i.e., $a(w,v) \neq a(v,w)$.

1.2 Discrete formulation

Our discrete formulation is based upon the weak formulation. In particular, we assume that we use K finite elements, T_h^k, $k = 1, ..., K$, and that the numerical solution is approximated as a first-order polynomial over each element (i.e., we use linear elements).

Mathematically, we define our finite-dimensional subspaces as

$$X_h = \{v \in X \mid v|_{T_h^k} \in \mathbb{P}_1(T_h^k),\ k = 1, ..., K\} \ . \tag{9}$$

The discrete problem can now be stated as: Find $u_h \in X_h$ such that

$$a(u_h, v) = l(v) \qquad \forall v \in X_h \ . \tag{10}$$

Hence, our numerical solution is piecewise linear, it is continuous across the elements, and it satisfies the prescribed homogeneous Dirichlet boundary conditions.

1.3 Algebraic formulation

As usual, we choose a nodal basis for X_h:

$$\forall v \in X_h\,, \qquad v(x) = \sum_{i=1}^{N} v_i\, \phi_i(x) \ . \tag{11}$$

$$\tag{12}$$

Note that $K = N+1$, and that $\forall v \in X_h,\ v(x_0) = v(x_{N+1}) = 0$.

Inserting this basis for u_h and v into the discrete formulation, we arrive at a set of algebraic equations

$$\underline{A}_h\, \underline{u}_h = \underline{F}_h \ , \tag{13}$$

where

$$u_h(x) = \sum_{i=1}^{N} u_{hi}\, \phi_i(x) \tag{14}$$

and

$$\underline{u}_h = [u_{h1}, u_{h2}, ..., u_{hN}]^T \ . \tag{15}$$

Because we are using a nodal basis, the unknowns represent the numerical solution at the *internal* global points x_i, $i = 1, ..., N$.

We assemble the global matrix $\underline{A}_h$ by adding up all the elemental contributions. The element matrix $\underline{A}_h^k$ associated with the bilinear form (7) can be expressed as

$$\underline{A}_h^k = \frac{\kappa}{h^k} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{U}{2} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \ . \tag{16}$$

Here, h^k is the length of element T_h^k, $k = 1, ..., K$.

The (local) grid Peclet number is defined as

$$P_g = Pe_g^k = \frac{U\,h^k}{\kappa} \ . \tag{17}$$

Note that the grid Peclet must be thought of as a *locally* defined quantity for a nonuniform grid. We know that, on a uniform grid, the numerical solution can exhibit oscillations when the grid Peclet number is greater than 2. In order to avoid having to resolve a thin boundary layer, but still being able to resolve the outer solution, it is common to use upwinding. In the finite element context, this is achieved by adding a controlled amount of diffusion in the streamwise direction. In particular, in $I\!R^1$, instead of using the physical thermal conductivity κ, we use the modified conductivity

$$\tilde{\kappa} = \kappa + U\,\frac{h^k}{2} \tag{18}$$

on each element. This corresponds to a modified diffusivity

$$\tilde{\varepsilon} = \varepsilon + \frac{h^k}{2} \ . \tag{19}$$

The boundary layer thickness will thus be $\tilde{\varepsilon}$ instead of ε. For a fixed discretization, the numerical solution will have a boundary layer of no less than $\frac{h^k}{2}$ even if $\varepsilon \to 0$. In this case, the numerical error will be $\mathcal{O}(1)$ in the vicinity of the boundary layer. On the other hand, we obtain a stable solution. If f varies slowly, the outer solution will typically be well resolved.

1.4 Theoretical analysis

We will now try to predict the oscillations that can occur when we use linear finite elements with no upwinding.

First, from (6) and (10), and using the fact that X_h is a subspace of X, we obtain the consistency property

$$a(u - u_h, v) = 0 \qquad \forall v \in X_h \ . \tag{20}$$

Next, we derive that, for all $v \in X$,

$$\begin{aligned} a(v,v) &= \int_0^1 (\kappa\, v_x v_x \,+\, U\, v_x v)\, dx \\ &= \kappa \int_0^1 v_x^2 \,+\, U \int_0^1 \frac{1}{2}\frac{d}{dx}(v^2)\, dx \\ &= \kappa \mid v \mid_{H^1}^2 \,+\, \frac{U}{2}\,(v^2(1) - v^2(0)) \\ &= \kappa \mid v \mid_{H^1}^2 \ . \end{aligned}$$

Through this last result, we have shown that the bilinear form a is coercive also in the non-symmetric case. It immediately follows that

$$a(u - u_h, u - u_h) = \kappa \mid u - u_h \mid^2_{H^1} \quad . \tag{21}$$

In the following, we will focus on the error in the semi-norm. We recall that, due to the Dirichlet boundary conditions, the semi-norm and the H_1-norm are equivalent in our case.

We now rewrite the bilinear form by using integration by parts on the convection term, as well as the homogeneous boundary conditions, to arrive at

$$a(w, v) = \int_0^1 (\kappa\, w_x v_x + U\, w_x v)\, dx \tag{22}$$

$$= \int_0^1 (\kappa\, w_x v_x - U\, w v_x)\, dx \quad . \tag{23}$$

We can the write

$$\begin{aligned} a(u - u_h, u - u_h) &= a(u - u_h, u - v_h) \qquad \forall v_h \in X_h \\ &= \int_0^1 (\kappa\,(u - u_h)_x (u - v_h)_x - U\,(u - u_h)(u - v_h)_x)\, dx \\ &\leq \kappa \mid u - u_h \mid_{H^1} \mid u - v_h \mid_{H^1} + U \parallel u - u_h \parallel_{L^2} \mid u - v_h \mid_{H^1} \\ &\leq \kappa \mid u - u_h \mid_{H^1} \mid u - v_h \mid_{H^1} + U\, C\, h \mid u - u_h \mid_{H^1} \mid u - v_h \mid_{H^1} \\ &= (\kappa + U\, C\, h) \mid u - u_h \mid_{H^1} \mid u - v_h \mid_{H^1} \quad . \end{aligned}$$

Here, the first equality follows from the consistency property (20), the first inequality follows from the Schwarz inequality, while the second inequality follows from bounding the L^2-error by the error in the semi-norm, that is,

$$\parallel u - u_h \parallel_{L^2} \leq C\, h \mid u - u_h \mid_{H^1} \tag{24}$$

where C is a constant independent of h.

Combining this last result with (21), we obtain that

$$\kappa \mid u - u_h \mid^2_{H^1} = a(u - u_h, u - u_h) \tag{25}$$

$$\leq (\kappa + C\, U\, h) \mid u - u_h \mid_{H^1} \mid u - v_h \mid_{H^1} \quad . \tag{26}$$

Hence, we finally arrive at the result

$$\mid u - u_h \mid_{H^1} \leq (1 + \frac{C\, U\, h}{\kappa}) \mid u - v_h \mid_{H^1} \quad \forall v_h \in X_h \quad . \tag{27}$$

We can also write this in terms of the grid Peclet number as

$$\mid u - u_h \mid_{H^1} \leq (1 + C\, P_g) \inf_{v_h \in X_h} \mid u - v_h \mid_{H^1} \quad . \tag{28}$$

From this last result, we see that the numerical solution u_h (or the gradient of the solution) may degrade compared to the best approximation result if $C\,P_g > 1$. If the constant C is $\mathcal{O}(1)$, u_h may become non-optimal for $P_g > 1$. This is, indeed, what we observe. By adding numerical diffusion, the Galerkin procedure will again give close to "optimal" results, but to a modified problem.

Recall that, when $U = 0$ (i.e., in the pure diffusion case), the error estimate reads

$$| u - u_h |_{H^1} = \inf_{v_h \in X_h} | u - v_h |_{H^1} \quad . \tag{29}$$

Exercise 1. Find the analytical solution for the problem (1)-(3) in the case of no convection (i.e., $U = 0$) and when $f = 1$. Plot the solution.

Exercise 2. Find the analytical solution for the problem (1)-(3) when $U = 1$ and $f = 1$. Plot the solution for different values of the Peclet number.