TMA4220 Finite Element Method
Fall 2016

Exercise set 1

1 Given a symmetric positive definite bilinear form $a(\cdot,\cdot)$ and linear form $F(\cdot)$, show that the problem: find $u_h \in H_0^1(\Omega)$ such that

$$a(u,v) = F(v), \quad \forall v \in H_0^1(\Omega)$$

has a unique solution if it exist (*Hint: consider two solutions u_1 and u_2 and show that they must be equal*).

2 Let $V = H_0^1(0,1)$, and take $a : V \times V \to \mathbb{R}$ and $F : V \to \mathbb{R}$ defined in the following way:

$$F(v) = \int_0^1 (-1-4x)v(x)\mathrm{d}x, \quad a(u,v) = \int_0^1 (1+x)u'(x)v'(x)\mathrm{d}x$$

a) Show that the bilinear form $a(\cdot,\cdot)$ is continuous and coercive and that the problem "find $u_h \in V$ such that $a(u,v) = F(v)$" has a unique solution by the Lax Milgram theorem.

b) Verify that this solution is $u(x) = x^2 - x$.

3 **a)** For which $\alpha \in \mathbb{R}$ does the function $f(x) := |x|^\alpha$ lie in $L^2([-1,1])$? What about $L^2([1,\infty))$? What about $L^2(B_1(0))$, where $B_1(0) = \{x \in \mathbb{R}^2 : |x| < 1\}$ is the unit ball in $\mathbb{R}^2$?

b) If $D \subset \mathbb{R}$ is a closed, bounded subset of $\mathbb{R}$ and $f \in C^0(D)$, show that $f \in L^2(D)$.

c) Let $\Omega \subset \mathbb{R}$ be some open interval. A *weak derivative* of a function $u : \Omega \to \mathbb{R}$ is a function $v : \Omega \to \mathbb{R}$ such that

$$\int_\Omega u(x)\phi'(x)\,dx = -\int_\Omega v(x)\phi(x)\,dx$$

for every $\phi \in C_c^\infty(\Omega)$, the set of infinitely differentiable functions with compact support in Ω. Show that the weak derivative (if it exists) is unique. Show that if u is continuously differentiable (i.e. $u \in C^1(\Omega)$), then $\frac{du}{dx}$ is its weak derivative.

d) Let

$$f_1(x) := \begin{cases} x & \text{if } 0 < x < 1 \\ 1 & \text{if } 1 \le x < 2, \end{cases} \qquad f_2(x) := \begin{cases} x & \text{if } 0 < x < 1 \\ 2 & \text{if } 1 \le x < 2 \end{cases}$$

for $x \in \Omega := (0,2)$. Show that $f_1, f_2 \in L^2(\Omega)$. Show that $f_1 \in H^1(\Omega)$ by finding its weak derivative, and that $f_1 \notin H^2(\Omega)$. Show that $f_2 \notin H^1(\Omega)$.

4 Consider the homogenuous dirichlet problem

$$\begin{aligned} -u''(x) &= 1, \quad x \in [0,1] \\ u(0) &= 0 \\ u(1) &= 0 \end{aligned}$$

and it's corresponding Galerkin problem:

Find $u_h \in X_h^1$ such that

$$\begin{aligned} a(u_h, v_h) &= F(v), \quad \forall v_h \in X_h^1 \\ a(u,v) &= \int_0^1 u'(x)v'(x)\mathrm{d}x \\ F(v) &= \int_0^1 v(x)\mathrm{d}x \\ X_h^1 &= \{v \in C^0([0,1]) : v|_{K_j} \in \mathbb{P}_1 \forall K_j \in \mathcal{T}_h\} \end{aligned}$$

Can be solved by the following `Matlab` code:

```
n = 20;                % number of nodal points
x = linspace(0,1,n); % nodal points
A = zeros(n);          % system matrix
b = zeros(n,1);        % right-hand side
h = diff(x);           % element size
for el=1:n-1           % element loop
  k = el:el+1;
  A(k,k) = A(k,k) + [1,-1;-1,1]/h(el);
  b(k)   = b(k)   + h(el)/2;
end
A([1,n],:) = [];     % remove boundary conditions
A(:,[1,n]) = [];
b([1,n])   = [];
u = A \ b;           % solve system
```

a) How would you modify the code to instead solve the following mixed boundary value problem

$$\begin{aligned} -u''(x) &= 1, \quad x \in [0,1] \\ u(0) &= 0 \\ u'(1) &= 1 \end{aligned}$$

b) What is the exact solution to this problem? Plot your finite element solution and the exact solution in the same plot.