Norwegian University of Science and Technology
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TMA4220 Finite Element Method Fall 2016

Exercise set 2

1 Consider the equation

$$-u_{xx} = f(x), \quad 0 < x < 1, \quad u(0) = 0, \quad u(1) = 0,$$

which is solved with the finite element method with equidistant grid ($x_i = ih$, $h = 1/N$), and the basis functions

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h}, & \text{for } x_{i-1} \le x \le x_i, \\ \frac{x_{i+1} - x}{h} & \text{for } x_i \le x \le x_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

for $i = 1, 2, \dots, N-1$.

Denote the numerical solution u_h.

a) Solve this equation with $f(x) = x^4$.

b) Plot the exact solution and the computed solution. How do these compare on the gridpoints. How do they compare between gridpoints?

You may have noticed that the numerical and exact solution coincide on the gridpoints, that is

$$u(x_i) = u_h(x_i), \quad \text{for } i = 0, 1, 2, \dots, N.$$

We will now show that this is true for all $f(x)$. For $i = 0$ and $i = N$, there is nothing to prove (Why?), what remains is $i = 1, 2, \dots, N-1$.

c) Prove that $a(u - u_h, \varphi_i) = 0$ for $i = 1, 2, \dots, N-1$.

d) Prove that for all $v \in H^1([0,1]) \cap C^0([0,1])$ and $i = 1, 2 \dots, N-1$,

$$a(v, \varphi_i) = \frac{1}{h}(2v(x_i) - v(x_{i-1}) - v(x_{i+1})).$$

e) The $M \times M$ matrix

$$A = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}$$

is invertible for all M. Use this fact and the results of **c)** and **d)** to prove that $u(x_i) - u_h(x_i) = 0$ for $i = 1, 2 \dots, N-1$.

2 Given the Helmholtz problem

$$-u_{xx} + \sigma u = f \text{ on } (0,1),$$
$$u(0) = u(1) = 0.$$

where $\sigma > 0$ is a constant. Set up the weak form for this problem. Show that, when this problem is solved by a Galerkin method, using $V_h = \text{span}\ \{\varphi_i\}_{i=1}^{N}$, the discrete problem can be written as

$$(A + \sigma M)\mathbf{u} = \mathbf{f}.$$

Set up the matrix M for $V_h = X_h^1$ on a uniform grid.

$\boxed{3}$ Write a MATLAB program for solving the Helmholtz problem

$$-u_{xx} + \sigma u = f(x), \qquad 0 < x < 1, \qquad u(0) = u(1) = 0.$$

or, using the weak formulation

$$\text{find } u \in H_0^1(0,1) \text{ s.t. } \int_0^1 u_x v_x dx + \sigma \int_0^1 uv dx = \int_0^1 fv dx, \text{ for all } v \in H_0^1(0,1) \quad (1)$$

by the finite element method on X_h^2, using the algorithm outlined in the supplementary note.

To test your code, let $\sigma = 1$, $f = \sin(\pi x)$ in which case $u(x) = \sin(\pi x)/(1 + \pi^2)$.

Use for example $[0, 0.1, 0.25, 0.3, 0.4, 0.45, 0.5, 0.55, 0.6, 0.7, 0.8, 0.9, 1]$ for the partition of the elements.

As already pointed out in exercise 2, the discrete problem can be written as

$$(A + \sigma M)\mathbf{u} = \mathbf{b}. \qquad (2)$$

So the task is to set up the matrices A and M and the load vector $\mathbf{b}$, and solve the system. What you have to do is described in the following:

a) Preliminaries:
Set up the element matrices A_h^K and M_h^K, corresponding to contribution from element K to the first and second integrals of (1) resp.

b) Write a function computing integrals by the following quadrature formula:

$$\int_0^1 g(x)dx \approx \frac{1}{2}(g(c_1) + g(c_2)), \qquad c_{1,2} = \frac{1}{2} \pm \frac{\sqrt{3}}{6}.$$

This will be used for to approximate the contribution from an element to the load vector.

c) Assemble the prototype matrices $\tilde{A}_h$ and $\tilde{M}_h$ as well as the load vector $\tilde{\mathbf{b}}$.

d) Remove the rows and columns corresponding to the boundary conditions.

e) Solve (2), and plot the solution.