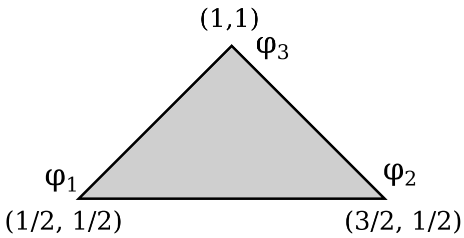
Norwegian University of Science and Technology Institute of mathematics	TMA4220 Finite Element Method Fall 2016 **Exercise set 4**

1 Consider the triangle with corners $(\frac{1}{2}, \frac{1}{2})$, $(1, 1)$ and $(\frac{3}{2}, \frac{1}{2})$. The linear functions can

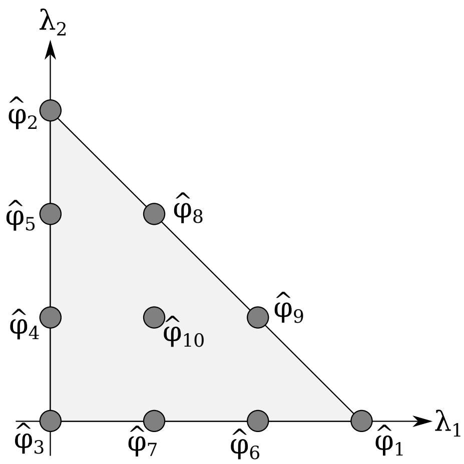


be written as

$$\varphi_i(x, y) = a_i x + b_i y + c$$

Find the expression for the three basis functions on this element in physical coordinates (x, y).

2 Consider the 10-node reference triangle of unit length: X_h^3. The *cubic* functions can

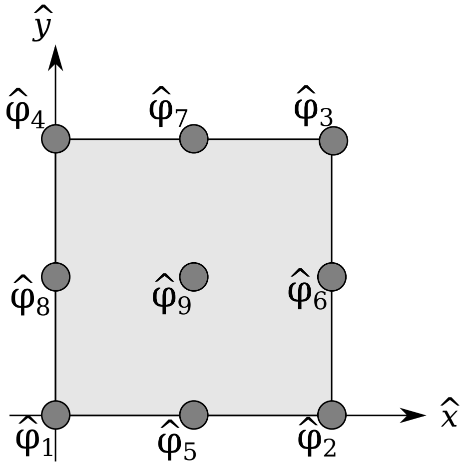


be written as

$$\hat{\varphi}_i(\lambda_1, \lambda_2, \lambda_3) = \sum_{0 \le i+j+k \le 3} a_{ijk} \lambda_1^i \lambda_2^j \lambda_3^k$$

Find the expression for the ten basis functions on this element in barycentirc (area) coordinates $(\lambda_1, \lambda_2, \lambda_3)$.

3 Consider the 9-node reference square of unit length: Y_h^2. The *quadratic* functions

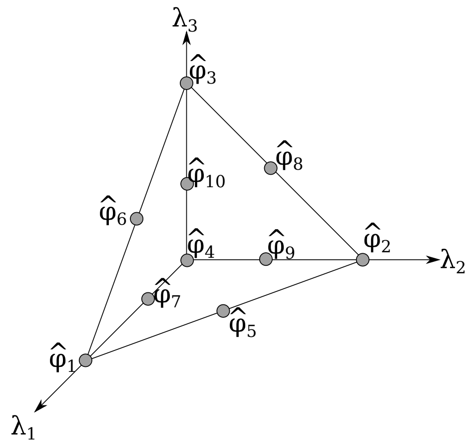


can be written as

$$\hat{\varphi}_i(\hat{x},\hat{y}) = \sum_{0\le i,j\le 2} a_{ij}\hat{x}^i\hat{y}^j$$

Find the expression for the nine basis functions on this element in reference coordinates $(\hat{x},\hat{y})$.

4 Consider the 10-node reference tetrahedron of unit length: X_h^2. The *quadratic* func-



tions can be written as

$$\hat{\varphi}_i(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = \sum_{0\le i+j+k+\ell\le 2} a_{ijk\ell}\lambda_1^i\lambda_2^j\lambda_3^k\lambda_4^\ell$$

Find the expression for the ten basis functions on this element in reference coordinates $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$.

5 Let the refernce element in task 2 be mapped to the physical element in task 1 by an affine mapping

$$\mathcal{F} : \mathbf{x}(\boldsymbol{\lambda}) = \mathbf{x}_1\lambda_1 + \mathbf{x}_2\lambda_2 + \mathbf{x}_3\lambda_3$$

where $\mathbf{x}_i$ are the three corners from task 1. Let the basis functions on the physical domain be the reference functions composed with the mapping: $\varphi_i = \hat{\varphi}_i \circ \mathcal{F}$. We will in the following compute the physical derivatives of the basis functions

$$\nabla\varphi_i = \begin{bmatrix} \frac{\partial\varphi_1}{\partial x} \\ \frac{\partial\varphi_1}{\partial y} \end{bmatrix}$$

a) Find the derivatives of the first basis function with respect to the barycentric coordinates $(\lambda_1, \lambda_2, \lambda_3)$:

$$\hat{\nabla}\hat{\varphi}_1 = \begin{bmatrix} \frac{\partial\hat{\varphi}_1}{\partial\lambda_1} \\ \frac{\partial\hat{\varphi}_1}{\partial\lambda_2} \\ \frac{\partial\hat{\varphi}_1}{\partial\lambda_3} \end{bmatrix}$$

In the following, it is convenient to use reference coordinates:

$$\begin{aligned} \lambda_1 &= \hat{x} \\ \lambda_2 &= \hat{y} \\ \lambda_3 &= 1 - \hat{x} - \hat{y} \end{aligned}$$

b) Compute the jacobian of the forward mapping $\mathcal{F}$:

$$J = \begin{bmatrix} \frac{\partial x}{\partial\hat{x}} & \frac{\partial x}{\partial\hat{y}} \\ \frac{\partial y}{\partial\hat{x}} & \frac{\partial y}{\partial\hat{y}} \end{bmatrix}$$

c) Compute the jacobian of the inverse mapping:

$$J^{-1} = \begin{bmatrix} \frac{\partial\hat{x}}{\partial x} & \frac{\partial\hat{x}}{\partial y} \\ \frac{\partial\hat{y}}{\partial x} & \frac{\partial\hat{y}}{\partial y} \end{bmatrix}$$

(Hint: J^{-1} is the same as the algebraic matrix inverse)

d)

$$\nabla\varphi_i = \begin{bmatrix} \frac{\partial\varphi_i}{\partial x} \\ \frac{\partial\varphi_i}{\partial y} \end{bmatrix} = G \begin{bmatrix} \frac{\partial\hat{\varphi}_i}{\partial\lambda_1} \\ \frac{\partial\hat{\varphi}_i}{\partial\lambda_2} \\ \frac{\partial\hat{\varphi}_i}{\partial\lambda_3} \end{bmatrix}$$

For some matrix $G \in \mathbb{R}^{2\times3}$. Using the chain-rule: what is the matrix G?

e) Determine the value of the physical derivatives of the first basis function $\nabla\varphi_1$ when evaluated at the element center $(\lambda_1, \lambda_2, \lambda_3) = (1/3, 1/3, 1/3)$.

f) Determine the value of the physical derivatives of the fourth basis function $\nabla\varphi_4$ when evaluated at the element center $(\lambda_1, \lambda_2, \lambda_3) = (1/3, 1/3, 1/3)$.