Problem Set 1

#1

Let u_1 & u_2 be solutions

$$a(u_1, v) = F(v) \quad \forall\, v \in H_0^1$$
$$a(u_2, v) = F(v) \quad \forall\, v \in H_0^1$$

Then

$$\begin{aligned} a(u_1 - u_2, v) &= a(u_1, v) - a(u_2, v) \\ &= F(v) - F(v) \\ &= 0 \end{aligned}$$

In particular for $v = u_1 - u_2$

$$a(u_1 - u_2, u_1 - u_2) = 0$$

Since $a(\cdot,\cdot)$ is positive definite

$$\Rightarrow u_1 - u_2 = 0$$
$$\underline{\underline{u_1 = u_2}}$$

#2

a) Continuous:

$$|a(u,v)| \underset{\text{definition}}{\leq} M \|u\|_V \cdot \|v\|_V \quad \forall\, u,v \in V$$

$$a(u,v) = \int_0^1 \underbrace{(1+x)}_{<2 \text{ on } [0,1]} u'(x) v'(x)\, dx \leq 2 \int_0^1 u'(x) v'(x)\, dx \underset{\text{Cauchy Schwarz}}{\leq} 2 \|u\|_{H^1} \cdot \|v\|_{H^1}$$

Coercive:

$$a(v,v) \underset{\text{definition}}{\geq} \alpha \|v\|_V^2 \quad \forall\, v \in V$$

$$a(v,v) = \int_0^1 \underbrace{(1+x)}_{>1 \text{ on } [0,1]} v'(x) v'(x)\, dx \geq \int_0^1 v'(x) v'(x)\, dx = |v|_{H^1(0,1)}^2 \underset{\text{Poincare inequality}}{\geq} \|v\|_{H^1(0,1)}^2$$

b) $u(x) = x^2 - x$

$u'(x) = 2x - 1$

$$a(u,v) = \int_0^1 (1+x)(2x-1)\,v'(x)\,dx$$

$$= \int_0^1 (2x^2+x-1)\,v'(x)\,dx$$

Integration by parts

$$\stackrel{\text{Integration by parts}}{=} \Big[(2x^2+x-1)\underbrace{v(x)}_{\substack{=0 \\ \text{since } v\in H_0^1}}\Big]_0^1 - \int_0^1 (2x^2+x-1)'\,v(x)\,dx$$

$$= \int_0^1 (-4x-1)\,v(x)\,dx = F(v)$$

#3

a) $L^2(-1,1)$:
$$\int_{-1}^1 (|x|^\alpha)^2\,dx = 2\int_0^1 x^{2\alpha}\,dx$$

$$= \left[\frac{1}{2\alpha+1}\,x^{2\alpha+1}\right]_0^1$$

$$= \underbrace{\frac{1}{2\alpha+1}}_{\substack{\text{finite} \\ \text{if } 2\alpha+1\neq 0}}\Big(1-\underbrace{0^{2\alpha+1}}_{\substack{\text{finite} \\ \text{if } 2\alpha+1\geq 0}}\Big)$$

$$\underline{\underline{|x|^\alpha \in L^2(-1,1) \iff \alpha > -1/2}}$$

$L^2(1,\infty)$
$$\int_1^\infty |x|^{2\alpha}\,dx = \left[\frac{1}{2\alpha+1}\,x^{2\alpha+1}\right]_1^\infty$$

$$= \lim_{x\to\infty} \frac{1}{2\alpha+1}\Big(\underbrace{x^{2\alpha+1}}_{\substack{\text{finite if} \\ 2\alpha+1<0}}-1\Big)$$

$$\underline{\underline{|x|^\alpha \in L^2(1,\infty) \iff \alpha < -1/2}}$$

$L^2(B_1(0))$:
$$\int_{B_1(0)} |x|^{2\alpha}\,dx = \int_{B_1(0)} (x^2+y^2)^\alpha\,dx\,dy$$

Change to polar coordinates

$$\int_0^{2\pi}\int_0^1 (r^2)^\alpha\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^1 r^{2\alpha}\,dr\,d\theta = 2\pi\left[\frac{1}{2\alpha+1} r^{2\alpha+1}\right]_0^1$$

$$= \frac{2\pi}{2\alpha+1}\Big(1 - \underbrace{0^{2\alpha+1}}_{\substack{\text{finite if} \\ 2\alpha+1>0}}\Big)$$

$$\underline{|x|^\alpha \in L^2(B_1(0)) \iff \alpha > -1/2}$$

b) Extreme value theorem:

If f is continuous on some interval $[a,b]$ then $f(x)$ takes some finite maximum value on this domain

$$\int_a^b |f(x)|^2\,dx < (b-a)\cdot \max_{x\in[a,b]} f(x) < \infty$$

$$\Rightarrow f \in L^2(D)$$

c) <u>Uniqueness</u>

Assume $v_1(x)$ & $v_2(x)$ are weak derivatives of $u(x)$

$$\int_\Omega (v_1(x)-v_2(x))\phi(x)\,dx = \int_\Omega v_1(x)\phi(x)\,dx - \int_\Omega v_2(x)\phi(x)\,dx$$

$$= \int_\Omega (u(x)-u(x))\phi'(x)\,dx$$

$$= 0$$

If $\int_\Omega (v_1(x)-v_2(x))\phi(x)\,dx = 0 \quad \forall\ \phi(x)\in C_0^\infty(\Omega)$

then $v_1(x)-v_2(x)=0$ almost everywhere

$$\Rightarrow v_1(x) = v_2(x)$$

<u>$u \in C^1(\Omega)$</u>

int. by parts

$$\int_\Omega u(x)\phi'(x)\,dx = \underbrace{\int_{\partial\Omega} u(x)\phi(x)\,dx}_{\substack{=0 \\ \text{since } \phi\in C_c^\infty(\Omega)}} - \int_\Omega \underbrace{u'(x)}_{=v(x)}\phi(x)\,dx$$

$$\underline{\underline{u'(x) = v(x)}}$$

d) $\int_0^2 [f_1(x)]^2 dx = \int_0^1 x^2 dx + \int_1^2 1 dx$

$= \frac{1}{3} + 1 < \infty$

$\int_0^2 (f_2(x))^2 dx = \int_0^1 x^2 dx + \int_1^2 2 dx < \infty$

$\Rightarrow f_1, f_2 \in L^2(0,2)$

$$\int_0^2 f_1(x)\phi'(x)\,dx = \int_0^1 x\phi'(x)\,dx + \int_1^2 1\phi'(x)\,dx$$

$$= [x\phi(x)]_0^1 - \int_0^1 1\,\phi(x)\,dx + [\phi(x)]_1^2 - \int_1^2 0\cdot\phi(x)dx$$

$$= \phi(1) + \underbrace{\phi(2)}_{\substack{=0 \\ \text{since } \phi\in C_c^\infty(\Omega) \\ \text{vanish on } \partial\Omega}} - \phi(1) - \int_0^1 1\,\phi(x)\,dx - \int_1^2 0\,\phi(x)dx$$

$v(x)$

$$v(x) = \begin{cases} 1 & x\in[0,1) \\ 0 & x\in[1,2) \end{cases}$$ is the weak derivative of $f_1(x)$

$$\int_0^2 f_2(x)\,\phi'(x)\,dx = \int_0^1 x\phi'(x)\,dx + \int_1^2 2\phi'(x)\,dx$$

$$= [x\phi(x)]_0^1 + [2\phi(x)]_1^2 - \int_0^1 1\cdot\phi(x)dx - \int_1^2 0\cdot\phi(x)dx$$

$$= \underbrace{\phi(1) + 2\phi(2) - 2\phi(1)}_{\neq 0} \quad \text{——} \; " \; \text{——}$$

f_2 has no weak derivative

#4

a) $-u''(x) = 1$

$$-\int_0^1 u''(x)\, v(x)\, dx = \int_0^1 1 \cdot v(x)\, dx$$

$$\int u'(x)\, v'(x)\, dx - [u'(x)\, v(x)]_0^1 = \int_0^1 v(x)\, dx$$

Let $v(x)$ satisfy the <u>dirichlet</u> condition $v(0)=0$

$$\int u'(x)\, v'(x)\, dx = \int_0^1 v(x)\, dx + \underbrace{u'(1)}_{=1} \cdot v(1)$$

Find $u_h \in V$ s.t

$$a(u_h, V_h) = F(v_h) \quad \forall\ v_h \in V_h$$

$$a(u,v) = \int_0^1 u'(x)\, v'(x)\, dx$$

$$F(v) = \int_0^1 v(x)\, dx + v(1)$$

$$V_h = \{ v \in H^1(0,1) : v(0) = 0 \}$$

<u>Changes</u>

1. $v(1)$ is no longer 0 $\forall\ v \in V_h$
2. $F(v)$ has changed (only affects $F(\varphi_n)$)

```
n=20;
x=linspace(   );
⋮

   b(k) = b(k) + h(el)/2;
end
b(end) = b(end) + 1;
A(1,:)= [];
A(:,1)= [];
b(1)  = [];
u     = A\b;
```