Exercise Set III

(TMA4220 2016)

#1

a) False

$v_1 \in S$, $v_2 \in S$

$v_3 = v_1 + v_2 \quad v_3(1/2) = v_1(1/2) + v_2(1/2) \neq 1$

$\Rightarrow v_3 \notin S$

b) True

$$F(\lambda v + \mu w) = \int_0^1 x(\lambda v + \mu w)\,dx = \lambda \int_0^1 xv\,dx + \mu \int_0^1 xw\,dx$$

$$= \lambda F(v) + \mu F(w)$$

c) False

$w = 1 \in H^1 \quad |w|_{H^1}^2 = \int_\Omega |\nabla w|^2 d\Omega = \int_\Omega 0^2 d\Omega = 0$

d) $$\|v\|_{L^2(0,1)}^2 = \int_0^1 (x^{3/4})^2 dx = \left[\frac{2}{5} x^{5/2}\right]_0^1 = \frac{2}{5} < \infty$$

$$|v|_{H^1(0,1)}^2 = \|v_x\|_{L^2}^2 = \int_0^1 \left(\frac{3}{4} x^{-1/4}\right)^2 dx$$

$$= \frac{9}{16} \int_0^1 x^{-1/2} dx = \frac{9}{16}\left[2x^{1/2}\right]_0^1 = \frac{9}{8} < \infty$$

$$|v|_{H^2(0,1)}^2 = \|v_{xx}\|_{L^2}^2 = \int_0^1 \left(-\frac{3}{16} x^{-5/4}\right)^2 dx$$

$$= \frac{9}{256} \int_0^1 x^{-5/2} dx \not< \infty$$

$v \in L^2(0,1)$ True

$v \in H^1(0,1)$ True

$v \in H^2(0,1)$ False

e) $v = e^{-10x}$

$$|v|_{H^1(0,1)} = \left(\int_0^1 v_x^2 \, dx\right)^{1/2} = \left(\int_0^1 (-10e^{-10x})^2 dx\right)^{1/2}$$

$$|v|_{H^2(0,1)} = \left(\int_0^1 v_{xx}^2 \, dx\right)^{1/2} = \left(\int_0^1 (-100e^{-10x})^2 dx\right)^{1/2}$$

False

#2

a) $\nabla^4 u = f$

$$\nabla^4 u \cdot v = f \cdot v$$

$$\int_\Omega \nabla^4 u \cdot v \, d\Omega = \int_\Omega f v \, d\Omega$$

$$-\int_\Omega \nabla^3 u \cdot \nabla v \, d\Omega + \int_{\partial\Omega} \underbrace{\cancel{v}}_{=0} \cdot \frac{\partial(\nabla^2 u)}{\partial n} d\gamma = \int_\Omega f \cdot v \, d\Omega$$

$$-\int_\Omega \nabla^3 u \cdot \nabla v \, d\Omega = \int_\Omega f v \, d\Omega$$

$$\int_\Omega \nabla^2 u \cdot \nabla^2 v \, d\Omega - \int_{\partial\Omega} \nabla^2 u \underbrace{\cancel{\frac{\partial v}{\partial n}}}_{=0} d\gamma = \int_\Omega f v \, d\Omega$$

Need second derivatives to exist:
$u \in H^2$

Weak form:
find $u \in H^2_0(\Omega)$ s.t

$$a(u,v) = F(v) \quad \forall v \in H^2_0(\Omega)$$

$$a(u,v) = \int_\Omega \nabla^2 u \, \nabla^2 v \, d\Omega$$

$$F(v) = \int_\Omega f \cdot v \, d\Omega$$

To use Lax-Milgram, we need $a(\cdot,\cdot)$ to be

continuoues:

$$a(u,v) \leq M \|u\|_V \|v\|_V$$

coercive:

$$a(v,v) \geq \alpha \|v\|_V^2$$

with $V = H_0^2(\Omega)$

and $$\|v\|^2_{H_0^2(\Omega)} = \underbrace{\int_\Omega v^2 \, d\Omega}_{\|v\|^2_{L^2(\Omega)}} + \underbrace{\int_\Omega v_x^2 + v_y^2 \, d\Omega}_{|v|^2_{H^1(\Omega)}} + \underbrace{\int_\Omega v_{xx}^2 + v_{xy}^2 + v_{yx}^2 + v_{yy}^2 \, d\Omega}_{|v|^2_{H^2(\Omega)}}$$

$$\begin{aligned} a(v,v) &= \int_\Omega \nabla^2 v \cdot \nabla^2 v \, d\Omega \\ &= \int_\Omega (v_{xx} + v_{yy})^2 \, d\Omega \\ &= \int_\Omega v_{xx}^2 + v_{yy}^2 \, d\Omega + 2\int_\Omega v_{xx} v_{yy} \, d\Omega \end{aligned}$$

$$= -2 \int_\Omega v_{xxy} v_y \, d\Omega$$

$$= 2 \int_\Omega v_{xy} v_{xy} \, d\Omega$$

Integration by parts

$$\begin{aligned} a(v,v) &= \int_\Omega v_{xx}^2 + v_{yy}^2 + 2v_{xy}^2 \, d\Omega \\ &= \|\nabla^2 v\|_{L^2(\Omega)} = |v|_{H^2(\Omega)} \geq C \|v\|_{H^2(\Omega)} \end{aligned}$$

Poincaré Inequality

To prove continuous we use Cauchy-Schwartz

$$a(u,v) = \int_\Omega \nabla^2 u \, \nabla^2 v \, d\Omega \leq C \left(\left(\int_\Omega |\nabla^2 u|^2 d\Omega \right)^{1/2} \left(\int_\Omega |\nabla^2 v|^2 d\Omega \right)^{1/2} \right)$$

$$= C \, \| \nabla^2 u \|_{L^2(\Omega)} \, \| \nabla^2 v \|_{L^2(\Omega)}$$

$$\leq C \, \| u \|_{H^2(\Omega)} \, \| v \|_{H^2(\Omega)}$$

b) $-\nabla^2 u + a \nabla u = f$

$$\int_\Omega -\nabla^2 u v + a \nabla u \cdot v \, d\Omega = \int_\Omega f v \, d\Omega$$

$$\int_\Omega \nabla u \nabla v \, d\Omega - \int_{\partial\Omega} \underbrace{v \frac{\partial u}{\partial n}}_{=0} d\gamma + \int_\Omega a \cdot \nabla u \cdot v \, d\Omega = \int_\Omega f v \, d\Omega$$

Weak form: find $u \in H_0^1(\Omega)$ s.t.

$$a(u,v) = F(v)$$

where

$$a(u,v) = \int_\Omega \nabla u \nabla v \, d\Omega + \int_\Omega v a \nabla u \, d\Omega$$

$$F(v) = \int_\Omega f v \, d\Omega$$

Need to show $a(\cdot,\cdot)$ coercive

$$a(v,v) = \int_\Omega |\nabla v|^2 d\Omega + \underbrace{\int_\Omega v a \nabla v \, d\Omega}_{\text{Divergence theorem}}$$

$$\int_{\partial\Omega} \underbrace{v^2}_{=0} a \cdot n \, d\gamma - \int_\Omega v \cdot \underbrace{\nabla(va)}_{= \nabla v \cdot a + v \underbrace{\nabla a}_{=0}} d\Omega$$

Divergence free a

$$\Rightarrow \int_\Omega v a \nabla v \, d\Omega = - \int_\Omega v \cdot \nabla v a \, d\Omega$$

$$2 \int v a \nabla v \, d\Omega = 0$$

$$a(v,v) = \int_\Omega |\nabla v|^2 d\Omega = |v|_{H^1(\Omega)} \geq C \|v\|_{H^1(\Omega)}$$

Poincaré Inequality

Continuous:

$$a(u,v) = (\nabla u, \nabla v)_{L^2} + (va, \nabla u)_{L^2}$$

Cauchy Schwartz $\leq \underbrace{(\nabla u, \nabla u)^{1/2}_{L^2}}_{|u|_{H^1}} (\nabla v, \nabla v)^{1/2}_{L^2} + (va, va)^{1/2}_{L^2} (\nabla u, \nabla u)_{L^2}$

$$= |u|_{H^1} \left(|v|_{H^1} + (va, va)^{1/2}_{L^2} \right)$$

$$\leq |u|_{H^1} \left(|v|_{H^1} + \max_{x \in \Omega}(a(x)) \, (v,v)^{1/2}_{L^2} \right)$$

$$= \underbrace{|u|_{H^1}}_{< \|u\|_{H^1}} \Big(\underbrace{|v|_{H^1} + C \|v\|_{L^2}}_{< C^* \|v\|_{H^1}} \Big) \qquad C^* = \max(1, C)$$

$$\Rightarrow a(u,v) \leq C^* \|u\|_{H^1} \|v\|_{H^1}$$