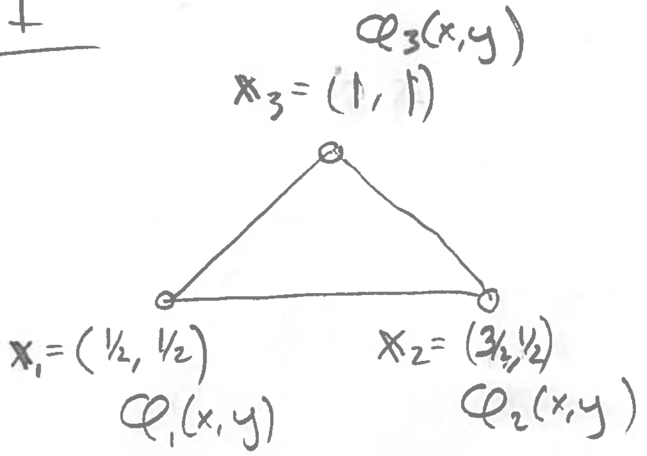
Solution Set IV

(TMA4220 Fall 2016)

#1



$$\varphi_i(\mathbf{x}_j) = \delta_{ij}$$

$$\varphi_i(x,y) = a_i x + b_i y + c$$

$$\varphi_1(1/2, 1/2) = 1/2\, a_1 + 1/2\, b_1 + c_1 = 1$$

$$\varphi_1(3/2, 1/2) = 3/2\, a_1 + 1/2\, b_1 + c_1 = 0$$

$$\varphi_1(1,1) = a_1 + b_1 + c_1 = 0$$

$$\underbrace{\begin{bmatrix} 1/2 & 1/2 & 1 \\ 3/2 & 1/2 & 1 \\ 1 & 1 & 1 \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}}_{K^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I}$$

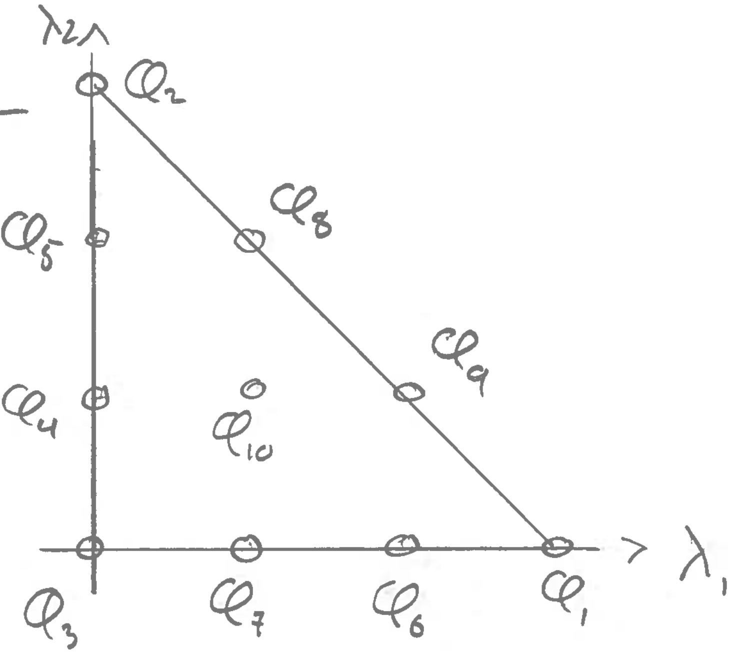
$$\Rightarrow \quad K^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 2 \\ 2 & 0 & -1 \end{bmatrix}$$

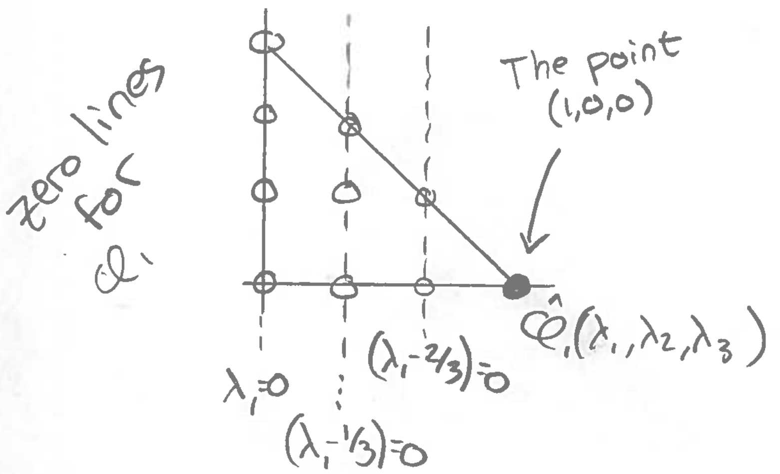
$$\underline{\underline{\varphi_1(x,y) = -x - y + 2}}$$

$$\underline{\underline{\varphi_2(x,y) = x - y}}$$

$$\underline{\underline{\varphi_3(x,y) = 2y - 1}}$$

#2





$$\hat{\varphi}_1^* = \lambda_1(\lambda_1 - 1/3)(\lambda_1 - 2/3)$$

Scaling gives

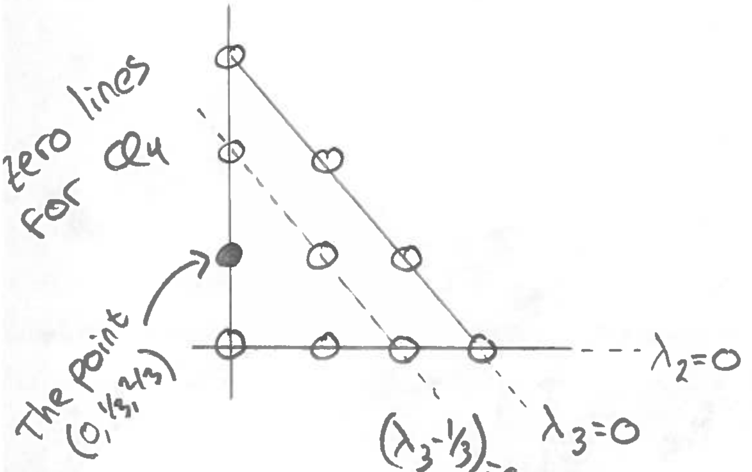
$$\hat{\varphi}_1^*(1,0,0) = 1 \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$$

$$\underline{\underline{\hat{\varphi}_1 = \frac{9}{2}\lambda_1(\lambda_1 - 1/3)(\lambda_1 - 2/3)}}$$

By symmetry we have

$$\underline{\underline{\hat{\varphi}_2 = \frac{9}{2}\lambda_2(\lambda_2 - 1/3)(\lambda_2 - 2/3)}}$$

$$\underline{\underline{\hat{\varphi}_3 = \frac{9}{2}\lambda_3(\lambda_3 - 1/3)(\lambda_3 - 2/3)}}$$

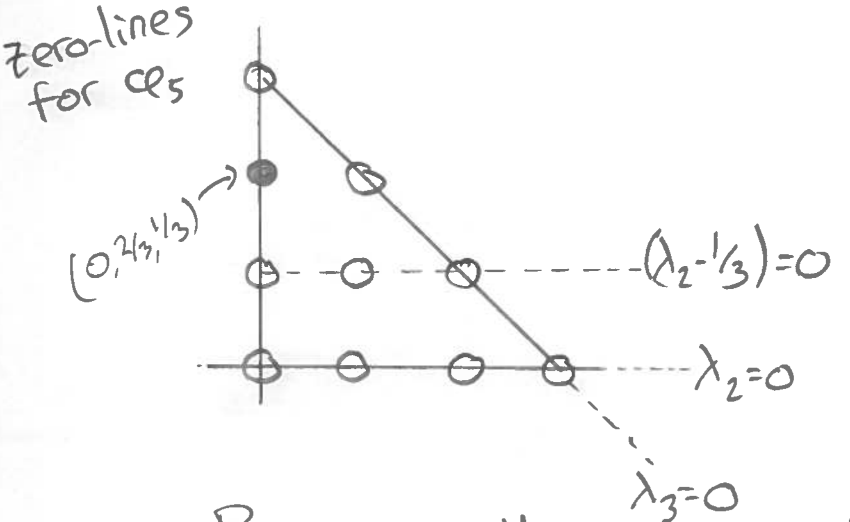


$$\hat{\varphi}_4^* = \lambda_2\lambda_3(\lambda_3 - 1/3)$$

Scaling gives

$$\hat{\varphi}_4^*(0, 1/3, 2/3) = \frac{1}{3}\,\frac{2}{3}\,\frac{1}{3}$$

$$\underline{\underline{\hat{\varphi}_4 = \frac{27}{2}\lambda_2\lambda_3(\lambda_3 - 1/3)}}$$



$$\hat{\varphi}_5^* = \lambda_2 \lambda_3 (\lambda_2 - 1/3)$$

$$\hat{\varphi}_5^*(0, 2/3, \tfrac{1}{3}) = \frac{2}{27}$$

$$\underline{\underline{\hat{\varphi}_5 = \frac{27}{2} \lambda_2 \lambda_3 (\lambda_2 - 1/3)}}$$

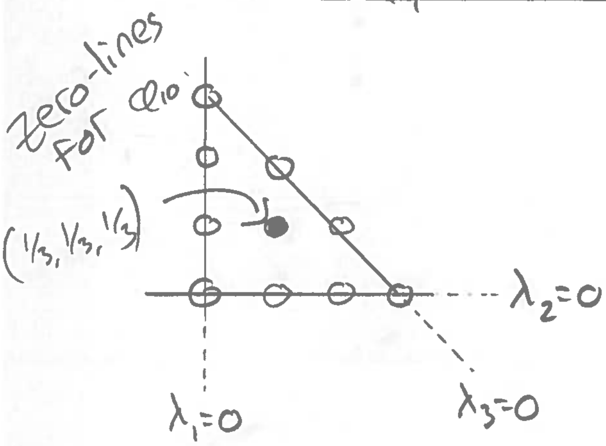
By symmetry we get

$$\underline{\underline{\hat{\varphi}_6 = \frac{27}{2} \lambda_3 \lambda_1 (\lambda_1 - 1/3)}}$$

$$\underline{\underline{\hat{\varphi}_7 = \frac{27}{2} \lambda_3 \lambda_1 (\lambda_3 - 1/3)}}$$

$$\underline{\underline{\hat{\varphi}_8 = \frac{27}{2} \lambda_1 \lambda_2 (\lambda_2 - 1/3)}}$$

$$\underline{\underline{\hat{\varphi}_9 = \frac{27}{2} \lambda_1 \lambda_2 (\lambda_1 - 1/3)}}$$

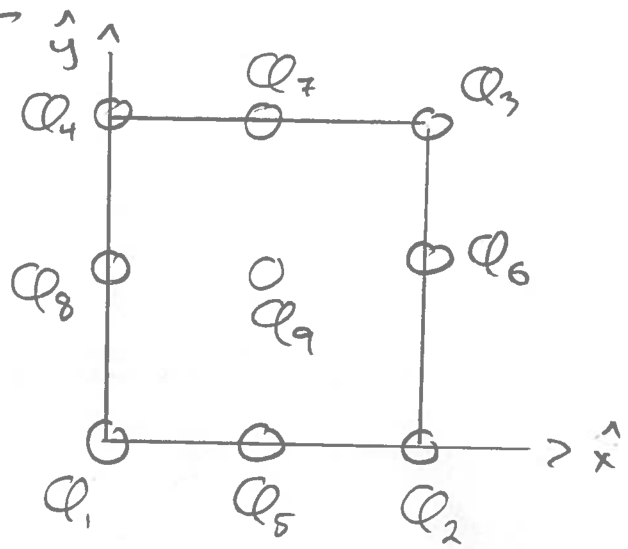


$$\hat{\varphi}_{10}^* = \lambda_1 \lambda_2 \lambda_3$$

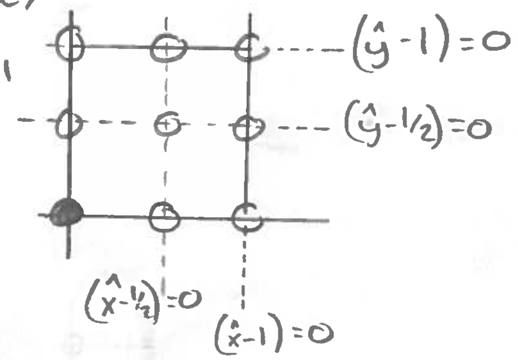
$$\hat{\varphi}_{10}^*(1/3, 1/3, 1/3) = \frac{1}{27}$$

$$\underline{\underline{\hat{\varphi}_{10} = 27 \lambda_1 \lambda_2 \lambda_3}}$$

#3



zero-lines for φ_1

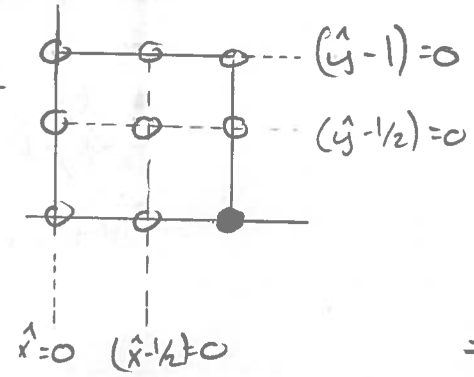


$$\hat{\varphi}_1^* = (\hat{x}-1/2)(\hat{x}-1)(\hat{y}-1/2)(\hat{y}-1)$$

$$\hat{\varphi}_1^*(0,0) = \frac{1}{4}$$

$$\underline{\underline{\hat{\varphi}_1 = 4(\hat{x}-1/2)(\hat{x}-1)(\hat{y}-1/2)(\hat{y}-1)}}$$

zero-lines for φ_2



$$\hat{\varphi}_2^* = \hat{x}(\hat{x}-1/2)(\hat{y}-1)(\hat{y}-1/2)$$

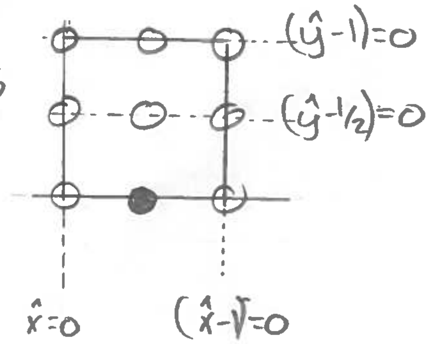
$$\hat{\varphi}_2^*(1,0) = \frac{1}{4}$$

$$\underline{\underline{\hat{\varphi}_2 = 4\hat{x}(\hat{x}-1/2)(\hat{y}-1)(\hat{y}-1/2)}}$$

$$\underline{\underline{\hat{\varphi}_3 = 4\hat{x}\hat{y}(\hat{x}-1/2)(\hat{y}-1/2)}}$$

$$\underline{\underline{\hat{\varphi}_4 = 4(\hat{x}-1)(\hat{x}-1/2)\hat{y}(\hat{y}-1/2)}}$$

zero-lines for φ_5



$$\hat{\varphi}_5^* = \hat{x}(\hat{x}-1)(\hat{y}-1/2)(\hat{y}-1)$$

$$\hat{\varphi}_5^*(1/2,0) = -\frac{1}{8}$$

$$\underline{\underline{\hat{\varphi}_5 = 8\hat{x}(1-\hat{x})(\hat{y}-1/2)(\hat{y}-1)}}$$

$$\hat{\varphi}_6 = 8\hat{x}\hat{y}(1-\hat{y})(\hat{x}-1/2)$$

$$\hat{\varphi}_7 = 8\hat{x}\hat{y}(\hat{y}-1/2)(1-\hat{x})$$

$$\hat{\varphi}_8 = 8\hat{y}(1-\hat{y})(\hat{x}-1/2)(\hat{x}-1)$$

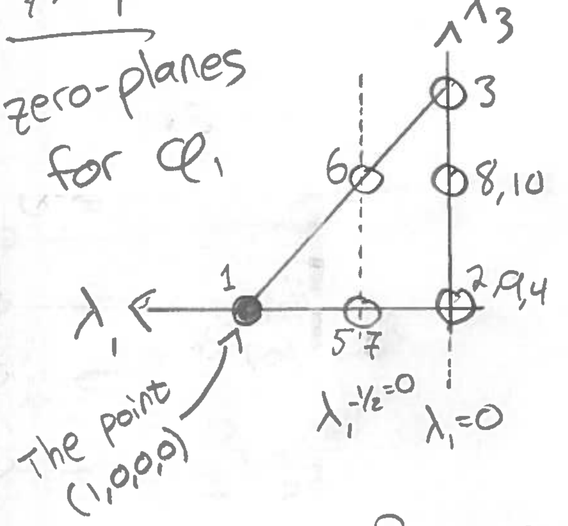
$$\hat{\varphi}_9^* = \hat{x}\hat{y}(1-\hat{x})(1-\hat{y})$$

$$\hat{\varphi}_9^*(1/2, 1/2) = \frac{1}{16}$$

$$\hat{\varphi}_9 = 16\hat{x}\hat{y}(1-\hat{x})(1-\hat{y})$$

#4

zero-planes for φ_1



$$\hat{\varphi}_1^* = \lambda_1(\lambda_1 - 1/2)$$

$$\hat{\varphi}_1^*(1,0,0,0) = 1/2$$

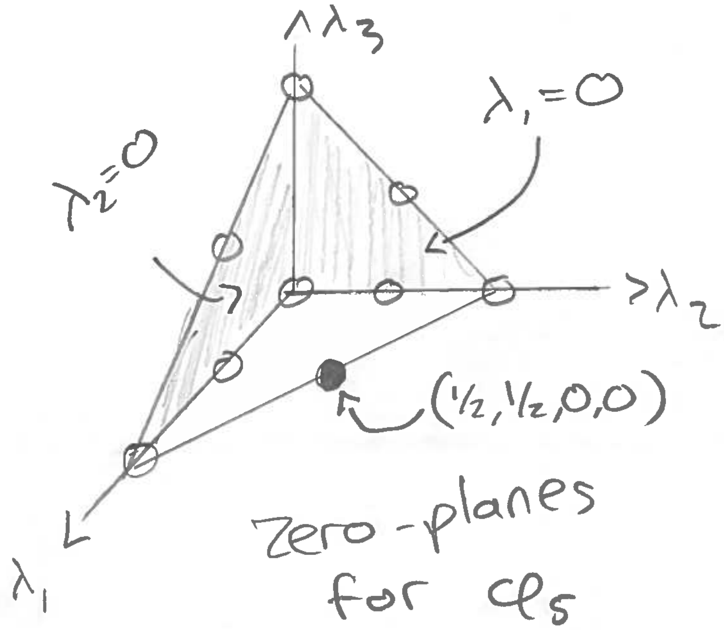
$$\hat{\varphi}_1 = 2\lambda_1(\lambda_1 - 1/2)$$

By symmetry

$$\hat{\varphi}_2 = 2\lambda_2(\lambda_2 - 1/2)$$

$$\hat{\varphi}_3 = 2\lambda_3(\lambda_3 - 1/2)$$

$$\hat{\varphi}_4 = 2\lambda_4(\lambda_4 - 1/2)$$



Zero-planes for φ_5

$$\hat{\varphi}_5^* = \lambda_1 \lambda_2$$

$$\hat{\varphi}_5^*(1/2, 1/2, 0, 0) = \frac{1}{4}$$

$$\underline{\underline{\hat{\varphi}_5 = 4\lambda_1\lambda_2}}$$

$$\underline{\underline{\hat{\varphi}_6 = 4\lambda_1\lambda_3}}$$

$$\underline{\underline{\hat{\varphi}_7 = 4\lambda_1\lambda_4}}$$

$$\underline{\underline{\hat{\varphi}_8 = 4\lambda_2\lambda_3}}$$

$$\underline{\underline{\hat{\varphi}_9 = 4\lambda_2\lambda_4}}$$

$$\underline{\underline{\hat{\varphi}_{10} = 4\lambda_3\lambda_4}}$$

#5

a) $\hat{\varphi}_1 = \frac{9}{2}\lambda_1(\lambda_1 - 1/3)(\lambda_1 - 2/3)$

$= \frac{9}{2}(\lambda_1^3 - \lambda_1^2 + \frac{2}{9}\lambda_1)$

$$\hat{\nabla}\hat{\varphi}_1 = \begin{bmatrix} \frac{9}{2}(3\lambda_1^2 - 2\lambda_1 + \frac{2}{9}) \\ 0 \\ 0 \end{bmatrix}$$

b) $\mathbb{X}_1 = (1/2, 1/2)$

$\mathbb{X}_2 = (3/2, 1/2)$

$\mathbb{X}_3 = (1, 1)$

$x = x_1\hat{x} + x_2\hat{y} + x_3(1-\hat{x}-\hat{y})$

$y = y_1\hat{x} + y_2\hat{y} + y_3(1-\hat{x}-\hat{y})$

$\frac{\partial x}{\partial \hat{x}} = x_1 - x_3 = 1/2 - 1$

$\frac{\partial x}{\partial \hat{y}} = x_2 - x_3 = 3/2 - 1$

$\frac{\partial y}{\partial \hat{x}} = y_1 - y_3 = 1/2 - 1$

$\frac{\partial y}{\partial \hat{y}} = y_2 - y_3 = 1/2 - 1$

$$J = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$

c) $|J| = (-\frac{1}{2})(-\frac{1}{2}) - \frac{1}{2}(-\frac{1}{2}) = \frac{1}{2}$

$$J^{-1} = 2\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ 1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{\partial \hat{x}}{\partial x} & \frac{\partial \hat{x}}{\partial y} \\ \frac{\partial \hat{y}}{\partial x} & \frac{\partial \hat{y}}{\partial y} \end{bmatrix}$$

d) $$\frac{\partial \varphi_i}{\partial x} = \frac{\partial \hat{\varphi}_i}{\partial \lambda_1}\frac{\partial \lambda_1}{\partial x} + \frac{\partial \hat{\varphi}_i}{\partial \lambda_2}\frac{\partial \lambda_2}{\partial x} + \frac{\partial \hat{\varphi}_i}{\partial \lambda_3}\frac{\partial \lambda_3}{\partial x}$$

$$\nabla \varphi_i = \underbrace{\begin{bmatrix} \frac{\partial \lambda_1}{\partial x} & \frac{\partial \lambda_2}{\partial x} & \frac{\partial \lambda_3}{\partial x} \\ \frac{\partial \lambda_1}{\partial y} & \frac{\partial \lambda_2}{\partial y} & \frac{\partial \lambda_3}{\partial y} \end{bmatrix}}_{G} \begin{bmatrix} \frac{\partial \hat{\varphi}_i}{\partial \lambda_1} \\ \frac{\partial \varphi_i}{\partial \lambda_2} \\ \frac{\partial \varphi_i}{\partial \lambda_3} \end{bmatrix}$$

Since $\lambda_1 = \hat{x}$ and we have $\frac{\partial \hat{x}}{\partial x} = -1$ from c)

$$\frac{\partial \lambda_1}{\partial x} = -1$$

Likewise $\frac{\partial \lambda_1}{\partial y} = -1$

$$\frac{\partial \lambda_2}{\partial x} = 1$$

$$\frac{\partial \lambda_2}{\partial y} = -1$$

$$\frac{\partial \lambda_3}{\partial x} = \frac{\partial}{\partial x}(1 - \hat{x} - \hat{y}) = -\frac{\partial \hat{x}}{\partial x} - \frac{\partial \hat{y}}{\partial x} = +1 - 1 = 0$$

$$\frac{\partial \lambda_3}{\partial y} = \frac{\partial}{\partial y}(1 - \hat{x} - \hat{y}) = -\frac{\partial \hat{x}}{\partial y} - \frac{\partial \hat{y}}{\partial y} = 1 + 1 = 2$$

$$G = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

e) $$\hat{\nabla}\hat{\varphi}_1\Big|_{(1/3,1/3,1/3)} = \begin{bmatrix} \frac{9}{2}\left(3\cdot\left(\frac{1}{3}\right)^2 - 2\,\frac{1}{3} + \frac{2}{9}\right) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ 0 \end{bmatrix}$$

$$\nabla\varphi_1\Big|_{(1/3,1/3,1/3)} = G\,\hat{\nabla}\hat{\varphi}_1\Big|_{(1/3,1/3,1/3)}$$

$$= \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -1/2 \\ 0 \\ 0 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}}}$$

f) $$\hat{\varphi}_4 = \frac{27}{2}\lambda_2\lambda_3(\lambda_3 - 1/3) \qquad \text{from \#2}$$

$$\hat{\nabla}\hat{\varphi}_4 = \frac{27}{2}\begin{bmatrix} 0 \\ \lambda_3(\lambda_3 - 1/3) \\ \lambda_2(2\lambda_3 - 1/3) \end{bmatrix}$$

$$\hat{\nabla}\hat{\varphi}_4\Big|_{(1/3,1/3,1/3)} = \frac{27}{2}\begin{bmatrix} 0 \\ 1/3\cdot 0 \\ 1/3\,(2/3 - 1/3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3/2 \end{bmatrix}$$

$$\nabla\varphi_4\Big|_{(1/3,1/3,1/3)} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 3/2 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}}$$