Exercises 1, TMA4225 Analysens grunnlag. Mandag 10:15 - 11:00, F4 (1.9.2003)

1. Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of sets. Define $\limsup_n E_n = \bigcap_{k=1}^{\infty} \bigcup_{n \geq k}^{\infty} E_n$. Show that $x \in \limsup_n E_n$ if and only if x lies in infinitely many E_n's.

2. Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers. Define $\liminf_n x_n = \sup_{k \geq 1}(\inf_{n \geq k} x_n)$. Show that $c = \liminf x_n$ if and only if the following two conditions are satisfied:

   (i) for all $\epsilon > 0$, there are finitely many x_n's in $(-\infty, c - \epsilon)$.

   (ii) for all $\epsilon > 0$, $(c - \epsilon, c + \epsilon)$ contains infinitely many x_n.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

   $$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \ (p \text{ and } q \text{ relatively prime}) \end{cases}$$

   Show that $f(x)$ is continuous at x, if $x \notin \mathbb{Q}$.

   Show that $f(x)$ is discontinuous at x, if $x \in \mathbb{Q}$.