1 **1**

**Problem 1**

Suppose that $X$ is a random variable with cumulative distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ x^3 & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

Calculate the following quantities and give your answers rounded three decimals after the decimal point.

**a)** The probability $\mathrm{P}(-2 < X < 0.5)$ = [ ] .

**b)** The expected value $\mathrm{E}(X)$ = [ ]

**c)** The variance $\mathrm{Var}(X)$ = [ ]

**d)** The standard deviation $\mathrm{Std}(X)$ = [ ]

**e)** The upper 5%-quantile of $X$ = [ ]

**Note: This problems should be answered only by filling out the above fields without handing in any handwritten text on paper.**

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Maximum marks: 10

2 **2**

**Problem 2**

Let $Y$ be a normally distributed random variable with an expected value of 10 and a standard deviation of 5. Find the following quantities and give your answers rounded to 3 decimals after the decimal point.

a) $P(Y < 0)$ =

b) $P(Y^2 = 100)$ =

c) $P(Y^2 - 20Y < 0)$ =

d) $E(Y^2)$ =

e) Let $f$ be the density function of Y. $f(15)$ =

**Note: This problems should be answered only by filling out the above fields without handing in any handwritten text on paper.**

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Maximum marks: 10

3 **3**

**Problem 3**

Suppose that X and Y has joint probability mass function $f(x, y)$ given by the following table:

| $x \backslash y$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0.1 | 0.2 | 0.1 |
| 1 | 0.4 | 0.2 | 0 |

Find the following quantities and give your answers rounded to three decimals after the decimal point.

**a)** $P(Y = 2) =$

**b)** $P(Y = 0|X = 1) =$

**c)** $E(Y) =$

**d)** $E(XY) =$

**e)** $\mathrm{corr}(5X - 2, 10Y + 3) =$

**Note: This problems should be answered only by filling out the above fields without handing in any handwritten text on paper.**

Maximum marks: 10

# 4 4

**Problem 4**

A producer of Lithium-Ion batteries deliverers batteries guaranteeing that their capacity is normally distributed with a mean of $\mu_0$ =2000mAh and a standard deviation of $\sigma_0$ =10mAh. A producer of mobile phones receives a trial batch consisting of a total of $N = 50$ batterier and measures the capacity $X_1, X_2, \ldots, X_n$ of the batteries in a sample os size $n = 25$ out of the total number of the $N = 50$ batteries in the trial batch. Assume that all battery capacities in the trial batch are independent. Furthermore, the observed mean battery capacity in the sample is given by $\bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i = 1995$mAh and the observed sample standard deviation is $s_n = 15.48$mAh.

**a)** Derive a 95%-confidence interval for $\mu$ on the basis of the observed sample under the assumption that the standard deviation of the measurements are as guaranteed by the battery producer and calculate the interval based on the above observations. Suppose that we instead carried out a two-sided hypothesis test of $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$ where $\mu_0 = 2000$ mAh using a significance level of $\alpha = 0.05$. What would the test conclusion have been?

**b)** Based on the observations, the mobile phone producer suspects that the standard deviation of the battery capacities is larger than guaranteed by the producer. Formulate this as a hypothesis test. Give the null and alternative hypothesis and a suitable test statistic given the above observed sample. Determine the critical value for a significance level of $\alpha = 0.05$. What is the conclusion of the test?

**c)** Let $F_{\chi^2_\nu}(x) = P(\chi^2_\nu \leq x)$ denote the cumulative distribution function of a chi-square random variable with $\nu$ degrees of freedom. Derive an expression for the power of the test in point b) in terms of $F_{\chi^2_\nu}(x)$ given that the true value of $\sigma = 15$mAh.

**d)** Let $\bar{X}_N = \frac{1}{N}\sum_{i=1}^{N} X_i$ denote the average battery capacity of all $N = 50$ batteries in the trial batch and let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ denote the sample average as before. Show that $\bar{X}_n - \bar{X}_N$ can be written as a linear combination of $X_1, X_2, \ldots, X_N$. What is the distribution of $\bar{X}_n - \bar{X}_N$? Find $E(\bar{X}_n - \bar{X}_N)$. Show that

$$\mathrm{Var}(\bar{X}_n - \bar{X}_N) = \frac{\sigma^2}{n}\left(1 - \frac{n}{N}\right).$$

Use this to derive a $(1-\alpha)$ prediction interval for the average capacity of the $N = 50$ batteries in the trial batch based on the measured capacities in the $n = 25$ observed sample.

Remember: Give your reasoning behind all your answers!

**It is recommended that you write your answer on the provided paper sheets.**

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Maximum marks: 40

5 **5**

**Problem 5**

A producer of electric car chargers produce a total of $m = 500$ chargers every day of production. Each day, as part of the quality control, a number of chargers are inspected until $r = 2$ faulty chargers have been found. In the following, we will look at $n = 10$ differenent days of production and let $Y_i$ denote the number of inspected chargers on day *i*, $i = 1, 2, \ldots, n$. Assume that different chargers are faulty independently of each other and with the same probability q. Assume also that $\sum_{i=1}^{10} y_i = 1601$. In the following, ignore the fact that each $Y_i$can take a value of a most m.

**a)** Explain why each $Y_i$ has a negative binomial distribution with parameters r and q. Calculate the probability $P(Y_i = 5)$ assuming that $q = 1/10$.

**b)** Show that the maximum likelihood estimator of the parameter q is given by $\hat{q} = 2n/\sum_{i=1}^{n} Y_i$ and compute an estimate of q given the above observations.

**c)** Explain why the distribution of $Y_i$ can be approximated by a normal distribution for large values of r. Use this to compute an estimate of the probability that $Y_i > m$ based on a normal approximation. If possible, also find an exact expression for the same proability and calcualte and the associated estimate.

Remember: Give your reasoning behind all your answers!

**It is recommended that you write your answer on the provided paper sheets.**

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Maximum marks: 30