Oppgave 1
Let $X$ and $Y$ be two independent normally distributed random variables. Assume that $X$ has expected value equal to 0 and standard deviation equal to 1, while $Y$ has expected value equal to 1 and standard deviation equal to 2.

Make a sketch of the probability densities for $X$ and $Y$ in one common plot. Determine the probabilities $P(X \leq 1.2)$, $P(Y > 2)$ and $P(X + Y \leq 2)$.

Oppgave 2
Let $A$ and $B$ be two events in a sample space $S$, where $P(A) = 0.2$, $P(B) = 0.5$ and $P(A \cup B) = 0.6$.

Are the events $A$ and $B$ disjoint? Are the events $A$ and $B$ independent?

Oppgave 3
The lifetime $T$ (measured in number of days (24 h)) of a new type of valve that is being considered for use on oil platforms in the North Sea, is to be checked. It is well known that the lifetime is influenced by for example the temperature and pressure where the valve is used, and also by the chemical composition of the oil that passes through the valves. Assume that the effect of these factors are measured as a stress factor $z$. It is also assumed that the cumulative distribution function for the lifetime $T$ is known by experience to be

$$F(t) = P(T \leq t) = \begin{cases} 1 - e^{-\frac{zt^2}{\theta}} & \text{for } t > 0, \\ 0 & \text{otherwise}, \end{cases}$$

where $\theta$ is an unknown parameter. The parameter $\theta$ is a characteristic feature for a particular type of valve, while $z$ describes the environment where the valve is used.

a) When $z = 2.0$ and $\theta = 2 \cdot 10^6$, determine the probabilities

$P(T > 1000)$ and $P(T > 2000|T > 1000)$.

Let $T_1, T_2$ and $T_3$ be the lifetimes of three valves that all operate under conditions where $z = 2.0$, and assume that $T_1, T_2$ and $T_3$ are independent random variables. When $\theta = 2 \cdot 10^6$, find the probability that at least two of the three lifetimes are greater than 1000 days.
b) Show that the probability density of $T$ is given by

$$f(t) = \begin{cases} \frac{2zt}{\theta}e^{-\frac{zt^2}{\theta}} & \text{for } t > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Make a sketch of $f(t)$ for $t \in [0, 3000]$ when $z = 2.0$ and $\theta = 2 \cdot 10^6$, and mark on the sketch the area that equals the probability $P(T > 1000)$.

Let a random variable $V$ be defined by

$$V = \frac{2zT^2}{\theta}.$$ 

c) Apply the transformation formula to show that $V$ is $\chi^2$ distributed with 2 degrees of freedom.

Use this result to show that $E(T^2) = \frac{\theta}{z}$ and $\text{Var}(T^2) = \left(\frac{\theta}{z}\right)^2$.

To check the quality of this type of valve, a random sample of $n = 10$ valves has been tested. Let $z_1, z_2, \ldots, z_n$ denote the stress factors under which these valves operate, and let $T_1, T_2, \ldots, T_n$ denote the corresponding lifetimes. It is assumed that $T_1, T_2, \ldots, T_n$ are independent random variables. The observed lifetimes are given in the following table:

<table>
<thead>
<tr>
<th>Valve $i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_i$</td>
<td>1.0</td>
<td>3.4</td>
<td>1.9</td>
<td>2.4</td>
<td>1.2</td>
<td>4.0</td>
<td>3.2</td>
<td>2.2</td>
<td>1.4</td>
<td>3.2</td>
</tr>
<tr>
<td>$t_i$</td>
<td>1297.2</td>
<td>834.2</td>
<td>1265.8</td>
<td>331.7</td>
<td>1937.8</td>
<td>727.6</td>
<td>869.6</td>
<td>746.7</td>
<td>1965.3</td>
<td>280.9</td>
</tr>
</tbody>
</table>

It has been calculated that $\sum_{i=1}^{n} z_i t_i^2 = 23\,287\,125$.

d) Derive the maximum likelihood estimator for $\theta$, and show that it can be written on the form

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} z_i T_i^2.$$ 

Is $\hat{\theta}$ unbiased? Find $\text{Var}(\hat{\theta})$.

e) Justify that

$$U = \sum_{i=1}^{n} \frac{2z_i T_i^2}{\theta} \sim \chi^2_{2n}.$$ 

Use this to derive a $(1 - \alpha) \cdot 100\%$ confidence interval for $\theta$. What is the confidence interval when the data are as given above and $\alpha = 0.05$?

On the basis of the observed data, one would also like to find a prediction interval for the lifetime, $T_0$, of a new valve which is to operate under conditions with a stress factor equal to $z_0$. To do this, one may start with the random variable

$$Y = n \cdot \frac{2T_0^2 z_0}{\sum_{i=1}^{n} \frac{2z_i T_i^2}{\theta}}.$$
where the numerator and the denominator in the fraction are independent random variables. The numerator and the denominator are both \( \chi^2 \) distributed, and they have 2 and \( 2n \) degrees of freedom, respectively. When \( n = 10 \), it can be shown that the probability density of \( Y \) becomes as shown in Figure 1. Note that some quantiles of this distribution are given to the right in the same figure.

**f)** Derive a 90\% prediction interval for \( T_0 \). What is the prediction interval when the data are as given above and \( z_0 = 3.0 \)?
Figure 2(a) shows the \( n = 42 \) observations, together with the estimated regression line. From the observations it is obtained that \( \sum_{i=1}^{n} (x_i - \bar{x})y_i = -12,163.6 \), \( \sum_{i=1}^{n} (x_i - \bar{x})^2 = 11,113.9 \) and \( \sum_{i=1}^{n} \left( y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i \right)^2 = 414,563.1 \).

For the remaining part of this problem you may (without proof) use that

\[
\frac{\hat{\beta}_1 - \beta_1}{\sqrt{\frac{S^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}}} \text{ is } t\text{-distributed with } n - 2 \text{ degrees of freedom.}
\]

It is required to investigate whether the observed data justifies the claim that the expected running time decreases with the number of push-ups.

**a)** Formulate this as a hypothesis test and construct a test for this purpose with significance level 5%.

What is the conclusion of the hypothesis test when the data are given as above?

**b)** The model we have been using in this problem assumes that the residuals, \( \varepsilon_i \), are normally distributed with mean value zero and with the same standard deviation \( \sigma \). Based on the plot in Figure 2(a), and the plot of the estimated residuals \( \hat{\varepsilon}_i = y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i \) in Figure 2(b), would you agree that this assumption is satisfied for the observed data? Justify your answer.

**Fasit**

1. 0.8849, 0.3085, 0.6736

3. a) 0.368, 0.050, 0.306 d) \( E[\hat{\theta}] = \theta, \text{ Var}[\hat{\theta}] = \theta^2/n \) e) [1 363 016, 4 856 037] f) [198.97, 1 645.92]
4. a) Do not reject $H_0$