Eksamen august 2010, oppgave 1

a - e) Pdf uke 9.

$X_1, X_2, \dots, X_n$ tilfeldig utvalg fra en log-normalfordeling med ~~tetthet~~ tetthet

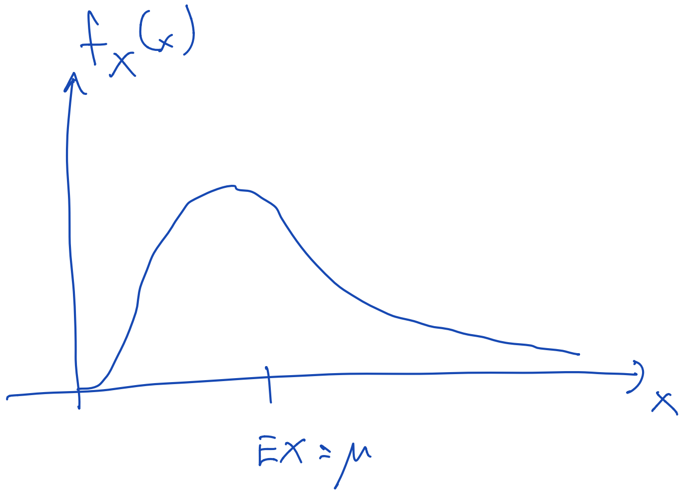
$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\tau x} e^{-\frac{1}{2\tau^2}(\ln x - \nu)^2}$$

Har vist at

$$Y = \ln X \sim N(\nu, \tau^2)$$

og

$$\mu = E(X) = e^{\nu + \tau^2/2}$$



f) Hvorfor er

$$\hat{\mu} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

en rimelig estimator av μ?

$$E(\hat{\mu}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n}\, n E(X_i) = \frac{1}{n} \cdot n \cdot \mu = \mu$$

d.v.s. $\hat{\mu}$ er forventningsrett for μ.

For oppgitte data blir

$$\hat{\mu} = \frac{1}{10}(57 + 38 + \ldots + 131) = 62.5$$

g) SME av ν er

$$\hat{\nu} = \frac{1}{n}\sum Y_i = \bar{Y} = \frac{1}{n}\sum_{i=1}^{n} \ln X_i = \frac{39.58}{10} = 3.958$$

og av τ^2

$$\hat{\tau}^2 = \frac{1}{n}\sum (Y_i - \bar{Y})^2 = \frac{3.236}{10} = 0.3260$$

Siden $\mu = e^{\nu + \tau^2/2}$ blir SME av μ

$$\mu^* = e^{\hat{\nu} + \hat{\tau}^2/2} = e^{3.958 + 0.3260/2} = 61.54.$$

Egenskaper

$$\hat{\nu} = \bar{Y} \sim N\left(\nu, \frac{\tau^2}{n}\right) \quad \text{(forventningsrett)}$$

$$E(\hat{\tau}^2) = \frac{n-1}{n} \cdot \tau^2 \quad \text{(forventningsfeil)}$$

μ^* er ikke nødvendigvis forventningsrett.

$$E(\hat\mu) = E\left(e^{\hat\nu + \hat\tau^2/2}\right)$$

$$= E\left(e^{\hat\nu}\, e^{\hat\tau^2/2}\right)$$

$$\frac{n\hat\tau^2}{\tau^2} = \frac{\sum (Y_i - \bar Y)^2}{\tau^2} \sim \chi^2_{n-1}$$

$$= \underline{E e^{\hat\nu}} \cdot E e^{\hat\tau^2/2}$$

$$= M_{\hat\nu}(1)\, \underline{E e^{\frac{\tau^2}{2n} \cdot \frac{n\hat\tau^2}{\tau^2}}}$$

tabell

$$= M_{\hat\nu}(1)\, M_{\chi^2_{n-1}}\left(\frac{\tau^2}{2n}\right)$$

$$= e^{\nu + \tau^2/(2n)} \left(\frac{1}{1 - 2\tau^2/(2n)}\right)^{\frac{n-1}{2}}$$

$$= \ldots \approx \mu e^{\frac{\tau^4}{4n}}$$

h) Kjenner $\tau^2 = \tau_0^2$. Nullhypotesen

$$H_0 : \mu \leq \mu_0$$

er ekvivalent med utsagnet

$$H_0 : e^{\nu + \tau_0^2/2} \leq \mu_0$$

$$\Updownarrow$$

$$\nu + \tau_0^2/2 \leq \ln \mu_0$$

$$\nu \leq \underbrace{\ln \mu_0 - \tau_0^2/2}_{\nu_0}$$

Tilsvarende blir

$$H_1 : \underline{\nu > \nu_0}$$

Under H_0 er testobservatør

$$\underline{Z} = \frac{\bar{Y} - \nu_0}{\tau_0/\sqrt{n}} \sim N(0,1)$$

Forkaster hvis

$Z > z_\alpha$ (øvre α-kvantil i standardnormalfordeling)

j) Konfidensintervall for μ?

$\tau = \tau_0^2$ kjent.

Da er

$$P\left(-z_{\alpha/2} < \overbrace{\underbrace{\frac{\bar{Y}-\nu}{\tau_0/\sqrt{n}}}_{\sim N(0,1)}}^{\text{pivotal størrelse}} < z_{\alpha/2}\right) = 1-\alpha$$

$$P\left(\bar{Y} - z_{\alpha/2}\tau_0/\sqrt{n} < \nu < \bar{Y} + z_{\alpha/2}\tau_0/\sqrt{n}\right) = 1-\alpha$$

slik at

$$\bar{Y} \pm z_{\alpha/2}\tau_0/\sqrt{n} = \frac{39{,}58}{10} \pm \frac{0.6}{10} = (3.58,\ 4.32)$$

er et $(1-\alpha)$-konf. int for ν.

Siden $\mu = e^{\nu + \tau_0^2/2}$ er et $(1-\alpha)$-konf.int for μ er

$$e^{\bar{Y} \pm z_{\alpha/2}\tau_0/\sqrt{n} + \tau_0^2/2}$$

$$= \left(e^{3.58 + 0.36/2}, e^{4.32 + 0.36/2}\right)$$

$$= (43.24, 90.96)$$

*) Prediksjonsintervall for ny observasjon X_0.

Lager først pred. intervall for

$$Y_0 = \ln X_0 \sim N(\nu, \tau_0^2)$$

Har at

$$E(\bar{Y} - Y_0) = 0$$

og

$$\mathrm{Var}(\bar{Y} - Y_0) = \mathrm{Var}(\bar{Y}) \overset{(-1)^2}{+} \mathrm{Var}(Y_0)$$

$$= \frac{\sigma_0^2}{n} + \sigma_0^2$$

$$= \sigma_0^2\left(1 + \frac{1}{n}\right)$$

slik at

$$P\left(-z_{\alpha/2} < \underbrace{\frac{\bar{Y} - Y_0}{\sigma_0\sqrt{1 + \frac{1}{n}}}}_{\sim N(0,1)} < z_{\alpha/2}\right) = 1 - \alpha$$

og

$$\bar{Y} \pm z_{\alpha/2}\,\sigma_0\sqrt{1 + \frac{1}{n}} = (2.72,\ 5.19)$$

blir et $(1-\alpha)$ – prediksjonsintervall for Y_0.

Siden $X_0 = e^{Y_0}$ er

$$P\left(\bar{Y} - z_{\alpha/2}\tau_0\sqrt{1+\tfrac{1}{n}} < Y_0 < \bar{Y} + z_{\alpha/2}\tau_0\sqrt{1+\tfrac{1}{n}}\right) = 1-\alpha$$

$$P\left(e^{\bar{Y} - z_{\alpha/2}\tau_0\sqrt{1+\frac{1}{n}}} < \underbrace{e^{Y_0}}_{X_0} < e^{\bar{Y} + z_{\alpha/2}\tau_0\sqrt{1+\frac{1}{n}}}\right) = 1-\alpha$$

slik at pred. int for X_0 blir

$$e^{\bar{Y} \pm z_{\alpha/2}\tau_0\sqrt{1+\frac{1}{n}}}$$

$$= \left(e^{2.72}, e^{5.19}\right)$$

$$= (15.25, 179.71)$$

Sanity check: Er dette rimelig i forhold til observerte data?