Eksamen juni 2010, oppgave 2

Levetid T til maskinkomponent har fetthet

$$f_T(t) = 2\lambda t e^{-\lambda t^2}, \quad t \geq 0$$

hvor $\lambda > 0$ er en ukjent parameter (sviktrate).

a) $F_T(t) = P(T \leq t)$

$$= \int_0^t f_T(u)\,du$$

$$= \int_0^t 2\lambda u e^{-\lambda u^2}\,du \qquad v = \lambda u^2 \qquad u = t \Rightarrow v = \lambda t^2$$

$$dv = 2\lambda u$$

$$= \int_0^{\lambda t^2} e^{-v}\,dv$$

$$= \left[-e^{-v}\right]_0^{\lambda t^2}$$

$$= -e^{-\lambda t^2} - (-1)$$

$$= 1 - e^{-\lambda t^2}$$

For $\lambda = 1.5 \cdot 10^{-3}$ blir

$$P(20 < T \leq 30) = F_T(30) - F_T(20)$$

$$= e^{-1.5 \cdot 10^{-3} \cdot 20^2} - e^{-1.5 \cdot 10^{-3} \cdot 30^2}$$

$$= 0{,}2896$$

b) Observer n levetider $T_1, \ldots, T_n$. Finn SME av λ.

Likelihood

$$L(\lambda) = \prod_{i=1}^n 2\lambda t_i e^{-\lambda t_i^2} = 2^n \lambda^n \left(\prod_{i=1}^n t_i\right) e^{-\lambda \sum t_i^2}$$

Log-likelihood

$$\ell(\lambda) = n \ln 2 + n \ln \lambda + \sum_{i=1}^n \ln t_i - \lambda \sum t_i^2$$

I maksimum er

$$\frac{\partial l}{\partial \lambda} = 0$$

$$\frac{n}{\lambda} - \sum t_i^2 = 0$$

$$\lambda = \frac{n}{\sum t_i^2}$$

SME av λ er

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} T_i^2}$$

c) $X \sim$ kji-kvadrat $(2n)$. Finn $E(X^{-1})$ og $E(X^{-2})$

$$f_X(x) = \frac{1}{2^n \Gamma(n)} x^{n-1} e^{-\frac{x}{2}}$$

$$E(X^{-1}) = \int_0^\infty x^{-1} f_X(x)\,dx$$

$$= \frac{1}{2^n \Gamma(n)} \int_0^\infty x^{n-1-1} e^{-\frac{x}{2}}\,dx$$

$$= \frac{2^{n-1}\Gamma(n-1)}{2^n \Gamma(n)} = \frac{1}{2(n-1)}$$

$$E(X^{-2}) = \int_0^\infty x^{-2} f_X(x)\,dx$$

$$= \frac{1}{2^n \Gamma(n)} \int_0^\infty x^{n-2-1} e^{-\frac{x}{2}}\,dx$$

$$= \frac{2^{n-2}\Gamma(n-2)}{2^n \Gamma(n)} = \frac{1}{4(n-1)(n-2)}$$

La $Y = 2\lambda T^2$. $\Leftrightarrow$ $T = \sqrt{\frac{Y}{2\lambda}}$. Da er

$$f_Y(y) = f_T(t(y)) \left| \frac{dt}{dy} \right|$$

$$= \cancel{2\lambda} \left(\frac{y}{\cancel{2\lambda}}\right)^{\frac{1}{2}} e^{-\lambda \frac{y}{2\lambda}} \frac{1}{2} y^{-\frac{1}{2}} \cancel{(2\lambda)^{-\frac{1}{2}}}$$

$$= \frac{1}{2} e^{-\frac{y}{2}}$$, d.v.s $Y \sim$ kji-kvadat (2)

$$E(\hat{\lambda}) = E\left(\frac{n}{\Sigma T_i^2}\right)$$

$$= E\left(\frac{2\lambda n}{\Sigma 2\lambda T_i^2}\right)$$

$$= 2\lambda n \, E\left(\frac{1}{X}\right)$$ hvor $X \sim$ kji-kvadat $(2n)$

$$= \frac{2\lambda n}{2(n-1)} = \frac{\lambda n}{n-1}$$

d.v.s. $\hat{\lambda}$ er forventningsskjev. Korrigert estimator

$$\lambda^* = \frac{n-1}{n} \hat{\lambda} = \frac{n-1}{\Sigma T_i^2}$$

forventingsrett.

$$Var(\lambda^*) = Var\left(\frac{n-1}{\Sigma T_i^2}\right)$$

$$= Var\left(\frac{(n-1)\, 2\lambda}{\Sigma\, 2\lambda T_i^2}\right)$$

$$= 4(n-1)^2 \lambda^2 \, Var\left(\frac{1}{X}\right)$$

$$= 4(n-1)^2 \lambda^2 \left[E(X^{-2}) - (EX^{-1})^2 \right]$$

$$= \cancel{4} (n-1)^2 \lambda^2 \left[\frac{1}{4(n-1)(n-2)} - \frac{1}{\cancel{4}(n-1)^2} \right]$$

$$= \lambda^2 \left[\frac{n-1}{n-2} - 1 \right]$$

$$= \lambda^2 \left[\frac{n-1-n+2}{n-2} \right] = \frac{\lambda^2}{n-2}$$

d) Konf. int. for λ:

Pivotal størrelse:

$$\sum_{i=1}^{n} 2\lambda T_i^2 \sim \text{kji-kvadrat}\,(2n)$$

$$\parallel$$

$$\frac{2\lambda n \sum T_i^2}{n} = \frac{2\lambda n}{\hat{\lambda}}$$

Dermed er

$$P\left(\chi^2_{1-\alpha/2,\,2n} < \frac{2\lambda n}{\hat{\lambda}} < \chi^2_{\alpha/2,\,2n} \right) = 1-\alpha$$

slik at

$$\left(\frac{\hat{\lambda}\, \chi^2_{1-\alpha/2,\,2n}}{2n}, \frac{\hat{\lambda}\, \chi_{\alpha/2,\,2n}}{2n} \right)$$

$$= \left(\frac{0.00407 \cdot 3.24}{10}, \frac{0.00407 \cdot 20.48}{10} \right) \qquad \hat{\lambda} = \frac{5}{2663^2 + \ldots + 11.55^2} = 0.00407$$

$$= (0.0013, 0.00834)$$

er et $(1-\alpha)$-konf. int. for λ.

e) Ønsker å teste

$$H_0: \lambda \leq \lambda_0 = 1.5 \cdot 10^{-3}$$

mot

$$H_1: \lambda > \lambda_0$$

Under H_0 er testobservatoren

$$W = 2\lambda_0 \sum T_i^2 \sim \text{kji-kvadrat}(2n)$$

$$\parallel$$

$$\frac{2n\lambda_0}{\hat{\lambda}}$$

NB! Hvis H_1 er sann vil W bli liten ($\hat{\lambda}$ stor).

Forkaster hvis $W < \chi^2_{1-\alpha, 2n} = \chi^2_{0.95, 10} = 3.94$

observert verdi

$$W = \frac{2n\, 1.5 \cdot 10^{-3}}{0.00407} = 3.6855$$

Forkaster H_0 til fordel for $H_1: \lambda < \lambda_0$.

f) Kasse med 5 komponenter.
Ønsker å stille garanti for hendelsen

$$\{T_1 > a \cap T_2 > a \cap \ldots \cap T_5 > a\}$$

$$= \{\bigcap_{i=1}^{5} T_i > a\}$$

Bestem a slik at

$$P(\text{Reklamasjon}) \leq 0.05$$

$$1 - P(\text{Ingen reklamasjon}) \leq 0.05$$

$$1 - P(\bigcap_{i=1}^{5} T_i > a) \leq 0.05$$

$$1 - \left(P(T_i > a) \right)^5 \leq 0.05$$

$$1 - \left(e^{-\lambda a^2} \right)^5 \leq 0.05$$

$$0.95 \leq e^{-5\lambda a^2}$$

$$\ln(0.95) \leq -5\lambda a^2$$

$$-\ln(0.95) \geq 5\lambda a^2$$

$$a \leq \sqrt{\frac{-\ln(0.95)}{5\lambda}}$$

$$= \sqrt{\frac{-\ln(0.95)}{5 \cdot 1.5 \cdot 10^{-3}}} = 2.615 \text{ uker}$$

g) a og λ som i punkt f). $n = 1000$ kasser hvorav det reklameres på U. Da er

$$U \sim \text{bin}(1000, 0.05)$$

$$P(U \leq 60) \overset{\text{kontinuitets korr.}}{\approx} P(U' < 60.5)$$

$$= P\left(\frac{U' - 50}{\sqrt{}} < \frac{60.5 - 50}{\sqrt{1000 \cdot 0.05 \cdot 0.95}} \right)$$

$$= \Phi\left(\frac{10.5}{\sqrt{50 \cdot 0.95}}\right) \approx 0.936$$

Eksamen mai 2012, oppgave 4

Grillfest:

a) A: Regn

B: Kaldt (under 15°C)

C: Ikke kaldt og ikke regn $\quad C = \bar{A} \cap \bar{B} = \overline{A \cup B}$

$P(A) = P(B) = 0.4, \quad P(A \cap B) = 0.2$

$P(A \cap B) > 0 \Rightarrow A \cap B$ mulig utfall $\Rightarrow A$ og B ikke disjunkte

$$P(C) = P(\overline{A \cup B})$$
$$= 1 - P(A \cup B) \quad \text{(komplementsregel)}$$
$$= 1 - \left(P(A) + P(B) - P(A \cap B)\right) \quad \text{(generell addisjonssetning)}$$
$$= 1 - P(A) - P(B) + P(A \cap B)$$
$$= 1 - 0.4 - 0.4 + 0.2$$
$$= 0.4$$

$A \cap C = A \cap (\bar{A} \cap \bar{B}) = \emptyset \Rightarrow A$ og C er disjunkte

$$P(C \mid A') = \frac{P(C \cap \bar{A})}{P(\bar{A})} \quad \text{(def. av betinget)}$$
$$= \frac{P(\bar{A} \cap \bar{B} \cap \bar{A})}{P(\bar{A})} \quad \text{(def av } C\text{)}$$
$$= \frac{P(\bar{A} \cap \bar{B})}{P(\bar{A})}$$

$$= \frac{P(C)}{1-P(A)} = \frac{0.4}{1-0.4} = 0.66\ldots$$

Regresjonsmodell:

$$Y_i = \alpha + \beta x_i + \varepsilon_i$$

Y_i: temp kl. 20

x_i: temp kl. 14

Tabell: SME av α og β gitt ved

$$\hat{\alpha} = \bar{Y} - \hat{\beta}\bar{x} \qquad \hat{\beta} = \frac{\sum (x_i - \bar{x}) Y_i}{\sum (x_i - \bar{x})^2}$$

b) Finn fordeling til $\hat{\alpha}$.

$\bar{Y}$, $\hat{\beta}$ og dermed $\hat{\alpha}$ er lineærkomb av $Y_1, \ldots, Y_n$ og dermed normalfordelt med

$$E(\hat{\alpha}) = E(\bar{Y} - \hat{\beta}\bar{x})$$

$$= E\bar{Y} - \bar{x}E(\hat{\beta})$$

$$= E\left(\frac{1}{n}\sum Y_i\right) - \bar{x}\beta \qquad (\hat{\beta} \text{ forventningsrett for } \beta)$$

$$= \frac{1}{n}\sum E(\alpha + \beta x_i + \varepsilon_i) - \bar{x}\beta$$

$$= \frac{1}{n}(n\alpha + \beta \sum x_i) - \bar{x}\beta$$

$$= \alpha + \beta\bar{x} - \bar{x}\beta$$

$$= \alpha$$

og

$$Var(\hat{\alpha}) = Var(\bar{Y} - \hat{\beta}\bar{x})$$

$$= \mathrm{Var}\,\bar{Y} - \bar{x}^2\,\mathrm{Var}(\hat{\beta}) \qquad (\bar{Y} \text{ og } \hat{\beta} \text{ uavh.})$$

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$$= \mathrm{Var}\Big(\underbrace{\tfrac{1}{n}\Sigma(\alpha+\beta x_i + \varepsilon_i)}_{\sigma^2/n}\Big) - \bar{x}^2 \underbrace{\frac{\sigma^2}{\Sigma(x_i-\bar{x})^2}}_{\text{oppsitt}}$$

$$= \frac{1}{n^2}\, n\,\mathrm{Var}(\varepsilon_i) - \frac{\sigma^2\,\bar{x}^2}{\Sigma(x_i-\bar{x})^2}$$

$$= \sigma^2\left(\frac{1}{n} - \frac{\bar{x}^2}{\Sigma(x_i-\bar{x})^2}\right)$$

c) Estimat av $E(Y_0)$ gitt $x_0 = 15°C$:

$$\hat{Y}_0 = \hat{\alpha} + \hat{\beta}x_0$$

$$= -6.57 + 1.43 \cdot 15$$

$$= 14.9\,°C$$

Prediksjonsintervall:

Utgangspunkt: $\hat{Y}_0 - Y_0$ er normalfordelt (lin. komb. av $Y_1, \dots, Y_n, \varepsilon_0$) med

$$E(\hat{Y}_0 - Y_0) = E(\hat{\alpha} + \hat{\beta}x_0 - \alpha - \beta x_0 - \varepsilon_0)$$

$$= 0 \quad \text{siden } E(\hat{\alpha}) = \alpha,\ E(\hat{\beta}) = \beta, \text{ og } E(\varepsilon_0) = 0$$

og

$$\mathrm{Var}(\hat{Y}_0 - Y_0) = \mathrm{Var}(\hat{\alpha} + \hat{\beta}x_0 - \alpha - \beta x_0 - \varepsilon_0)$$

funksjoner av $Y_1, \dots, Y_n$

uavh.

$$= \mathrm{Var}(\bar{Y} - \hat{\beta}\bar{x} + \hat{\beta}x_0 - \varepsilon_0)$$

$$= \mathrm{Var}(\bar{Y} + \hat{\beta}(x_0 - \bar{x}) - \varepsilon_0)$$

$$= \mathrm{Var}(\hat{Y}) + (x_0 - \bar{x})^2 \mathrm{Var}(\hat{\beta}) + \mathrm{Var}(\varepsilon_0) \quad (\text{uavh.})$$

$$= \sigma^2 \left(\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{\sum (x_i - \bar{x})^2} + 1 \right)$$

Dermed er

$$Z = \frac{\hat{Y} - Y_0}{\sigma \sqrt{1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{\sum (x_i - \bar{x})^2}}} \sim N(0,1)$$

Siden $V = \frac{S^2 (n-2)}{\sigma^2} \sim \chi^2_{n-2}$ (TMA4267)

og uavhengig av Z er

$$\left. T = \frac{Z}{\sqrt{V/(n-2)}} = \frac{\dfrac{\hat{Y} - Y_0}{\sigma \sqrt{1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{\sum (x_i - \bar{x})^2}}}}{\sqrt{\dfrac{S^2 (n-2)}{\sigma^2} \Big/ (n-2)}} = \frac{\hat{Y} - Y_0}{S \sqrt{1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{\sum (x_i - \bar{x})^2}}} \right\} \text{pivotal størrelse}$$

t-fordelt med $n-2$ frihetsgrader (teorem 8.5)

$$P\left(-t_{\alpha/2, n-2} < \frac{\hat{Y} - Y_0}{S \sqrt{1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{\sum (x_i - \bar{x})^2}}} < t_{\alpha/2, n-2} \right) = 1 - \alpha$$

slik at

$$\hat{Y} \pm t_{\alpha/2, n-2} \, S \sqrt{1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{\sum (x_i - \bar{x})^2}}$$

er et $(1-\alpha)$-prediksjonsintervall for Y_0 (temp kl. 20)

Innsetting: $1 - \alpha = 95\%$

$$14.9 \pm \overbrace{2.1}^{\text{tabell}} \cdot 2.04 \sqrt{1 + \frac{1}{20} + \frac{(15 - 15.5)^2}{510.17}}$$

$$= (10.5,\ 19.3) \quad (\text{Rimelig? Se figur})$$