Eksamen august 2010, oppgave 1

Nikkelinnhold X i jordprøver har sanns. tetthet

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\tau x} e^{-\frac{1}{2\tau^2}(\ln x - \nu)^2}, \quad x > 0$$

a) La $Y = \ln X$. Finn $f_Y(y)$. $\qquad Y = u(X) = \ln X \iff X = w(Y) = e^Y$

Metode 1: Trans. formel:

$$f_Y(y) = f_X(w(y)) \cdot |w'(y)|$$

$$= \frac{1}{\sqrt{2\pi}\,\tau e^y} \cdot e^{-\frac{1}{2\tau^2}(\ln e^y - \nu)^2} |e^y|$$

$$= \frac{1}{\sqrt{2\pi}\,\tau} e^{-\frac{1}{2\tau^2}(y-\nu)^2}$$

d.v.s. $\quad Y \sim N(\nu, \tau^2)$

Metode 2: Via kumulative

$$F_Y(y) = P(Y \leq y)$$

$$= P(\ln X \leq y)$$

$$= P(X \leq e^y)$$

$$= F_X(e^y)$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(e^y) = f_Y(e^y)\, e^y$$

b) Finn $F_X(x) = P(X \leq x)$

$$= P(e^Y \leq x)$$

$$= P(Y \leq \ln x) = F_Y(\ln x)$$

$$= P\Big(\underbrace{\frac{Y-\nu}{\tau}}_{\sim N(0,1)} \leq \frac{\ln x - \nu}{\tau}\Big)$$

$$= P\Big(Z \leq \frac{\ln x - \nu}{\tau}\Big)$$

$$= \Phi\Big(\frac{\ln x - \nu}{\tau}\Big)$$

c) Anta at $\nu = 1.0$ og $\tau = 0.8$.

$P(X_1 \cdot X_2 \leq 1.0) = ?$

$$= P(\ln(X_1 \cdot X_2) \leq \ln(1.0))$$

$$= P(Y_1 + Y_2 \leq 0)$$

$$= P\Big(\underbrace{\frac{Y_1 + Y_2 - 2\nu}{\sqrt{2\tau^2}}}_{\sim N(0,1)} \leq \frac{-2\nu}{\sqrt{2}\,\tau}\Big)$$

$$= \Phi\Big(-\frac{2 \cdot 1.0}{\sqrt{2}\; 0.8}\Big) =$$

d) $p = P(X < 2.72) = \Phi\Big(\frac{\ln 2.72 - 1.0}{0.8}\Big)$

$W \sim bin(5, p)$

$P(W \geq 4) = P(W=4) + P(W=5)$

e) Finn $\mu = EX$ og $\sigma^2 = Var(X)$

Siden $Y = \ln X \Leftrightarrow X = e^Y$ er

$$E(X^r) = E((e^Y)^r) = E(e^{rY})$$

$$= M_Y(r) \quad \text{(def av moment. gen. funksjon)}$$

Siden $Y \sim N(\nu, \tau^2)$ er

$$M_Y(t) = e^{\nu t + \frac{1}{2}\tau^2 t^2} \quad \text{(tabell)}$$

slik at

$$EX = M_Y(1) = e^{\nu + \frac{1}{2}\tau^2}$$

$$E(X^2) = M_Y(2) = e^{2\nu + 2\tau^2}$$

og

$$Var\, X = E(X^2) - (EX)^2$$

$$= e^{2\nu + 2\tau^2} - e^{2\nu + \tau^2}$$

$$= e^{2\nu}\left(e^{2\tau^2} - e^{\tau^2}\right)$$

Momentgenererende funksjoner

Anta at $Z \sim N(0,1)$. Da er

$$M_Z(t) = E e^{tZ} = E(g(Z))$$

$$= \int \underbrace{e^{tz}}_{g(z)} f_Z(z)\, dz$$

$$= \int \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2} + zt} \, dz$$

$$= \int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\underbrace{z^2 - 2tz + t^2}_{(z-t)^2} - t^2)} \, dz$$

$$= e^{t^2/2}$$

Anta at $Y \sim N(\mu, \sigma^2)$. Da kan Y representeres ved

$$Y = \mu + \sigma Z$$

og

$$M_Y(t) = E e^{tY} = E e^{t(\mu + \sigma Z)}$$

$$= e^{t\mu} E(e^{t\sigma Z})$$

$$= e^{t\mu} M_Z(t\sigma)$$

$$= e^{t\mu} e^{(t\sigma)^2/2}$$

$$= e^{\mu t + \sigma^2 t^2/2}$$

Anta at $X \sim \text{Poisson}(\lambda)$. Da er

$$
\begin{aligned}
M_X(t) &= Ee^{tX} \\
&= \sum_{x=0}^{\infty} e^{tx} f(x) \\
&= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!} \\
&= e^{-\lambda} e^{\lambda e^t} \underbrace{\sum_{x=0}^{\infty} \overbrace{\frac{(\lambda e^t)^x e^{-\lambda e^t}}{x!}}^{\text{punktsanns.-funksjon til Poisson}(\lambda e^t)}}_{=1} \\
&= e^{\lambda(e^t - 1)}
\end{aligned}
$$

Generellt:

$$M'_X(t) = \frac{d}{dt} Ee^{tX} = E \frac{d}{dt} e^{tX} = EXe^{tX} \Rightarrow M'_X(0) = EXe^{0X} = EX$$

$$M''_X(t) = \ldots = EX^2 e^{tX} \Rightarrow M''_X(0) = E(X^2)$$

Poisson eks. forts.:

$$M'_X(t) = e^{\lambda(e^t - 1)} \lambda e^t = \lambda e^{\lambda(e^t - 1) + t} \Rightarrow EX = M'_X(0) = \lambda$$

$$M''_X(t) = \lambda e^{\lambda(e^t - 1) + t}(\lambda e^t + 1) \Rightarrow E(X^2) = M''_X(0) = \lambda(\lambda + 1)$$

$$\text{Var } X = \lambda(\lambda + 1) - \lambda^2 = \lambda$$

Anta at $X_1 \sim \text{Poisson}(\lambda_1)$ og $X_2 \sim \text{Poisson}(\lambda_2)$. Da har $W = X_1 + Y_1$

$$
\begin{aligned}
M_W(t) &= Ee^{t(X_1 + X_2)} \\
&= Ee^{tX_1} e^{tX_2} \\
&= Ee^{tX_1} Ee^{tX_2} \\
&= M_{X_1}(t) M_{X_2}(t) \\
&= e^{\lambda_1(e^t - 1)} e^{\lambda_2(e^t - 1)} = e^{(\lambda_1 + \lambda_2)(e^t - 1)}
\end{aligned}
$$

d.v.s. $W = X_1 + X_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$

uavh. av

Anta at $Y_1 \sim N(\mu_1, \sigma_1^2)$ og $Y_2 \sim N(\mu_2, \sigma_2^2)$

La $W = Y_1 + Y_2$. Da er

$$M_W(t) = Ee^{tW} = E\left(e^{t(Y_1+Y_2)}\right)$$

$$= E\left(\underbrace{e^{tY_1}}\ \underbrace{e^{tY_2}}\right)$$

uavh.

$$= E\left(e^{tY_1}\right)E\left(e^{tY_2}\right)$$

$$= M_{Y_1}(t)\, M_{Y_2}(t)$$

$$= e^{\mu_1 t + \sigma_1^2 t^2/2}\ e^{\mu_2 t + \sigma_2^2 t^2/2}$$

$$= e^{(\mu_1+\mu_2)t + (\sigma_1^2+\sigma^2)^2 t^2/2}$$

Altså er $W \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

På tilsvarende måte kan det vises at Poisson-, Gamma-, neg. bin.-fordeling er _lukket_ under addisjon.

Ordningsvariable:

Utvalg $X_1, X_2, \dots, X_n$ iid fra tetthet/cdf $f_X(x)$, $F_X(x)$

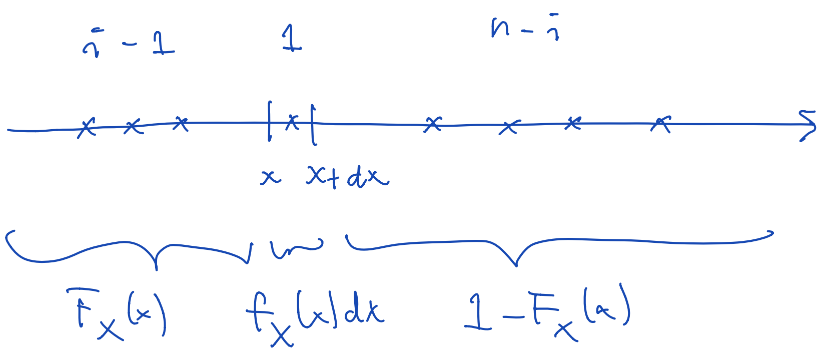
Minimum $X_{(1)} = \min(X_1, \dots, X_n)$ har ~~tetthet~~

$$\underbrace{f_{X_{(1)}}(x)\,dx}_{\substack{\text{sannsyn. for at} \\ X_{(1)} \in (x,\, x+dx)}} = \underbrace{n f_X(x)\,dx}_{\substack{\text{sannsyn.} \\ \text{for at én av} \\ n \text{ obs. havner} \\ \text{i intervallet} \\ (x,\, x+dx)}} \underbrace{\left(1 - F_X(x)\right)^{n-1}}_{\substack{n-1 \text{ andre} \\ \text{observasj.} \\ \text{større} \\ \text{enn } x}}$$

Maximum

$$f_{X_{(n)}}(x) = n f_X(x) \, F_X(x)^{n-1}$$

Tetthet til $X_{(i)}$: Multinomisk situasjon:

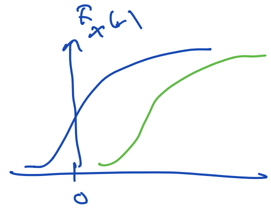


$$f_{X_{(i)}}(x)\,dx = \frac{n!}{(i-1)!\,1!\,(n-i)!} \left(F_X(x)\right)^{i-1} \left(f_X(x)\,dx\right)^{1} \left(1 - F_X(x)\right)^{n-i}$$

Generaliserer til simultantetthet til f.eks. $X_{(i)}, X_{(j)}$
(se wikipedia)

Eks.: Anta at $X_1, \dots, X_n$ er uavh. med kumulativ fordeling

$$F_X(x) = e^{-e^{-x}} \quad \text{(Standard Gumbel)}$$



Finn tetthet til $X_{(n)} = \max(X_1, \dots, X_n)$

Hver X_i har tetthet

$$f_X(x) = e^{-e^{-x}} (-e^{-x})(-1)$$

$$= e^{-x - e^{-x}}$$

$X_{(n)}$ har kumulativ tetthet

$$F_{X_{(n)}}(x) = P(X_{(n)} \le x)$$

$$= P(X_1 \le x \cap X_2 \le x \cap \dots \cap X_n \le x)$$

$$= F_X(x)^n$$

$$= \left(e^{-e^{-x}}\right)^n$$

$$= e^{-n e^{-x}}$$

$$= e^{-e^{-(x - \ln n)}}$$

$$f_{X_{(n)}}(x) = e^{-e^{-(x-\ln x)}} \left(-e^{-(x-\ln x)}\right)(-1)$$

$$= e^{-(x-\ln x) - e^{-(x-\ln x)}}$$

d.v.s Gumbel med lokasjonspar $\ln n$

Gumbel-fordeling er lukket når vi tar maksimum.