Oppgave 6

X: diameter (mm)

$X \sim N(\mu, \sigma = 0.5)$

$n = 10, \quad \bar{x} = 40.2$

a) 95% PI for X_0

X_0 antas uavh. av $X_1, \dots, X_{10}$

$X_0 \sim N(\mu, \sigma = 0.5)$

$X_i \sim N(\mu, \sigma = 0.5)$

$\Rightarrow \bar{X} \sim N(\mu, \frac{\sigma}{\sqrt{n}})$

$\bar{X} - X_0 \sim N(0, \sqrt{\frac{\sigma^2}{n} + \sigma^2})$

$$\Rightarrow \frac{\bar{X} - X_0}{\sigma\sqrt{\frac{1}{n}+1}} \sim N(0,1)$$

1. Stokastisk variabel

$$Z_1 = \frac{\bar{X} - X_0}{\sigma\sqrt{\frac{1}{n}+1}} \sim N(0,1)$$

2. kvantiler

$$P(\underbrace{z_{1-\frac{\alpha}{2}}}_{= -z_{\frac{\alpha}{2}} \text{ i norm.ford.}} \leq Z \leq z_{\frac{\alpha}{2}}) = 1-\alpha$$

3. Løse ulikhet mhp. X_0

$$-z_{\frac{\alpha}{2}} \leq \frac{\bar{X} - X_0}{\sigma\sqrt{\frac{1}{n}+1}}$$

$$-z_{\frac{\alpha}{2}}\sigma\sqrt{\frac{1}{n}+1} \leq \bar{X} - X_0$$

$$X_0 \leq \bar{X} + z_{\frac{\alpha}{2}}\sigma\sqrt{\frac{1}{n}+1}$$

$$\frac{\bar{X} - X_0}{\sigma\sqrt{\frac{1}{n}+1}} \leq z_{\frac{\alpha}{2}}$$

$$\bar{X} - z_{\frac{\alpha}{2}}\sigma\sqrt{\frac{1}{n}+1} \leq X_0$$

4. Sette sammen

$$P(\bar{X} - z_{\frac{\alpha}{2}}\sigma\sqrt{\frac{1}{n}+1} \leq X_0 \leq \bar{X} + z_{\frac{\alpha}{2}}\sigma\sqrt{\frac{1}{n}+1}) = 1-\alpha$$

5. intervall

$$[\bar{x} \pm z_{\frac{\alpha}{2}}\sigma\sqrt{\frac{1}{n}+1}]$$

Tall: 95% $\Rightarrow \alpha = 0.05$

$z_{\frac{\alpha}{2}} = 1.96$

$$40.2 \pm 1.96 \cdot 0.5 \cdot \sqrt{\frac{1}{10}+1}$$

$$= [39.17, 41.23]$$

b) Mindre

Oppgave 4

$$Y_i \sim N(\underbrace{\theta}_{\text{ukjent}} \overbrace{x_i}^{\text{kjent}}(1-x_i),\ \sigma^2 \overbrace{x_i}^{\text{kjent}})$$

↑ vanlig.

SME $\hat{\theta} = \dfrac{\sum Y_i(1-x_i)}{\sum x_i(1-x_i)^2}$

95% KI for θ

1. pivotal

$$E(\hat{\theta}) = E\left(\frac{\sum Y_i(1-x_i)}{\sum x_i(1-x_i)^2}\right)$$

$$= \frac{\sum E(Y_i)(1-x_i)}{\sum x_i(1-x_i)^2}$$

$$= \frac{\sum \theta x_i(1-x_i)(1-x_i)}{\sum x_i(1-x_i^2)}$$

$$= \frac{\theta \cancel{\sum x_i(1-x_i)^2}}{\cancel{\sum x_i(1-x_i)^2}} = \theta$$

$$\mathrm{Var}(\hat{\theta})$$

$$= \mathrm{Var}\left(\frac{\sum Y_i(1-x_i)}{\sum x_i(1-x_i)^2}\right)$$

$$= \frac{\sum \mathrm{Var}(Y_i)(1-x_i)^2}{(\sum x_i(1-x_i)^2)^2}$$

$$= \frac{\sum \sigma^2 x_i(1-x_i)^2}{(\sum x_i(1-x_i)^2)^2}$$

$$= \frac{\sigma^2 \cancel{\sum x_i(1-x_i)^2}}{(\sum x_i(1-x_i)^2)^{\cancel{2}}}$$

$$= \frac{\sigma^2}{\sum x_i(1-x_i)^2}$$

$$\Rightarrow \hat{\theta} \sim N(\theta, \sigma^2/\textstyle\sum x_i(1-x_i)^2)$$

$$Z = \frac{\hat{\theta} - \theta}{\sigma/\sqrt{\sum x_i(1-x_i)^2}} \sim N(0,1)$$

2.+3.+4. kvantiler, ulikhet, sette sammen

$$P(z_{1-\frac{\alpha}{2}} \le Z \le z_{\frac{\alpha}{2}}) = 1-\alpha$$

$$\uparrow = -z_{\frac{\alpha}{2}}$$

$$-z_{\frac{\alpha}{2}} \le \frac{\hat{\theta} - \theta}{\sigma/\sqrt{\sum x_i(1-x_i)^2}}$$

$$\theta \le \hat{\theta} + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{\sum x_i(1-x_i)^2}}$$

$$\frac{\hat{\theta} - \theta}{\sigma/\sqrt{\sum x_i(1-x_i)^2}} \le z_{\frac{\alpha}{2}}$$

$$\hat{\theta} - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{\sum x_i(1-x_i)^2}} \le \theta$$

$$P\left(\hat{\theta} - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{\sum x_i(1-x_i)^2}} \le \theta \le \hat{\theta} + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{\sum x_i(1-x_i)^2}}\right) = 1-\alpha$$

5. intervall

$$95\% \Rightarrow \alpha = 0.05 \Rightarrow z_{\frac{\alpha}{2}} = 1{,}96$$

$$\hat{\theta} \pm 1{,}96 \cdot \frac{\sigma}{\sqrt{\sum x_i(1-x_i)^2}}$$

Oppgave 5

X: # feilfrie komponenter (av n)

b) (1 min)

$X \sim$ Binomisk

- uavh. forsøk
- suksess (feilfri) eller ikke i hvert forsøk
- $P(\text{suksess}) = p = 0.9$ i alle forsøk
- $n = 20$ forsøk

c) $\hat{P} = \frac{X}{n}$, $\hat{p} = \frac{x}{n}$

$$\frac{X - np}{\sqrt{n\hat{p}(1-\hat{p})}} \approx N(0,1)$$

90 % KI for p

1.

$$Z = \frac{X - np}{\sqrt{n\hat{p}(1-\hat{p})}}$$

2. + 3. + 4.

$$P(-z_{\frac{\alpha}{2}} \le Z \le z_{\frac{\alpha}{2}}) = 1 - \alpha$$

$$-z_{\frac{\alpha}{2}} \le \frac{X - np}{\sqrt{n\hat{p}(1-\hat{p})}}$$

$$np \le X + z_{\frac{\alpha}{2}}\sqrt{n\hat{p}(1-\hat{p})}$$

$$p \le \frac{X}{n} + z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$\frac{X - np}{\sqrt{n\hat{p}(1-\hat{p})}} \le z_{\frac{\alpha}{2}}$$

$$X - z_{\frac{\alpha}{2}}\sqrt{n\hat{p}(1-\hat{p})} \le np$$

$$\frac{X}{n} - z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p$$

$$P\left(\frac{X}{n} - z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \frac{X}{n} + z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right) = 1 - \alpha$$

$\alpha = 0.1 \Rightarrow z_{\frac{\alpha}{2}} = 1.645$

Merk: $\frac{X}{n} = \hat{P}$

5.

$$\left[\hat{p} - z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$$

$n = 500$, $x = 470 \Rightarrow \hat{p} = 0.94$

$$0{,}94 \pm 1.645 \cdot \sqrt{\frac{0.94(1-0.94)}{500}}$$

$$= \underline{\underline{[0{,}923, 0{,}957]}}$$

I dette intervallet har vi 90 % tillit til at vi finner den ukjente andelen feilfrie komponenter

i produksjonen. Hvis vi gjentar forsøket med n=500 mange ganger vil i snitt 90 % av intervallene inneholde den samme (men ukjente) andelen.