Forelesning 19.02.2025

Tema: Momentgenererende funksjoner og sentralgrenseteoremet

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Kjikvadratfordeling

- La $X \sim \chi^2_\nu$.
  - $f_X(x) = \dfrac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\frac{\nu}{2}-1} e^{-x/2} \quad x \geq 0$

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- Definisjon: $M_X(t) = E[e^{tX}]$ og $E[h(X)] = \int_{-\infty}^{\infty} h(x) f_X(x) dx$
- Hint: Det blir et integral som er vanskelig å løse med vanlige regneregler. Kan du omskrive uttrykket så det har en kjent integrand med kjent verdi?

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$$\mathrm{E}(X) = \nu\ , \quad \mathrm{Var}(X) = 2\nu\ , \quad M_X(t) = \left(\frac{1}{1-2t}\right)^{\nu/2} \text{ for } t < \frac{1}{2}.$$

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Sum av uavhengige kjikvadratfordelte tilfeldige variabler

- La $X_1, X_2, \ldots, X_n$ være uavhengige og $X_i \sim \chi^2_{\nu_i}$.
  - Vi vet da at $M_{X_i}(t) = \dfrac{1}{(1-2t)^{\nu_i/2}}$.

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- La $Z \sim \chi^2_{\sum_{i=1}^{n} \nu_i}$.

2. Hva er da $M_Z(t)$?

- Hva kan vi si om Z og Y?

Binomisk fordeling

- La $X_1, X_2, \ldots, X_n$ være uavhengige og $X_i \sim f(x)$ med

$$f(x) = \begin{cases} p & x = 1, \\ 1 - p & x = 0, \\ 0 & \text{ellers} \end{cases}$$

Dvs. $P(X = 1) = p$ og $P(X = 0) = 1 - p$.

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1. Bestem $E[X]$ og $\text{Var}[X]$.

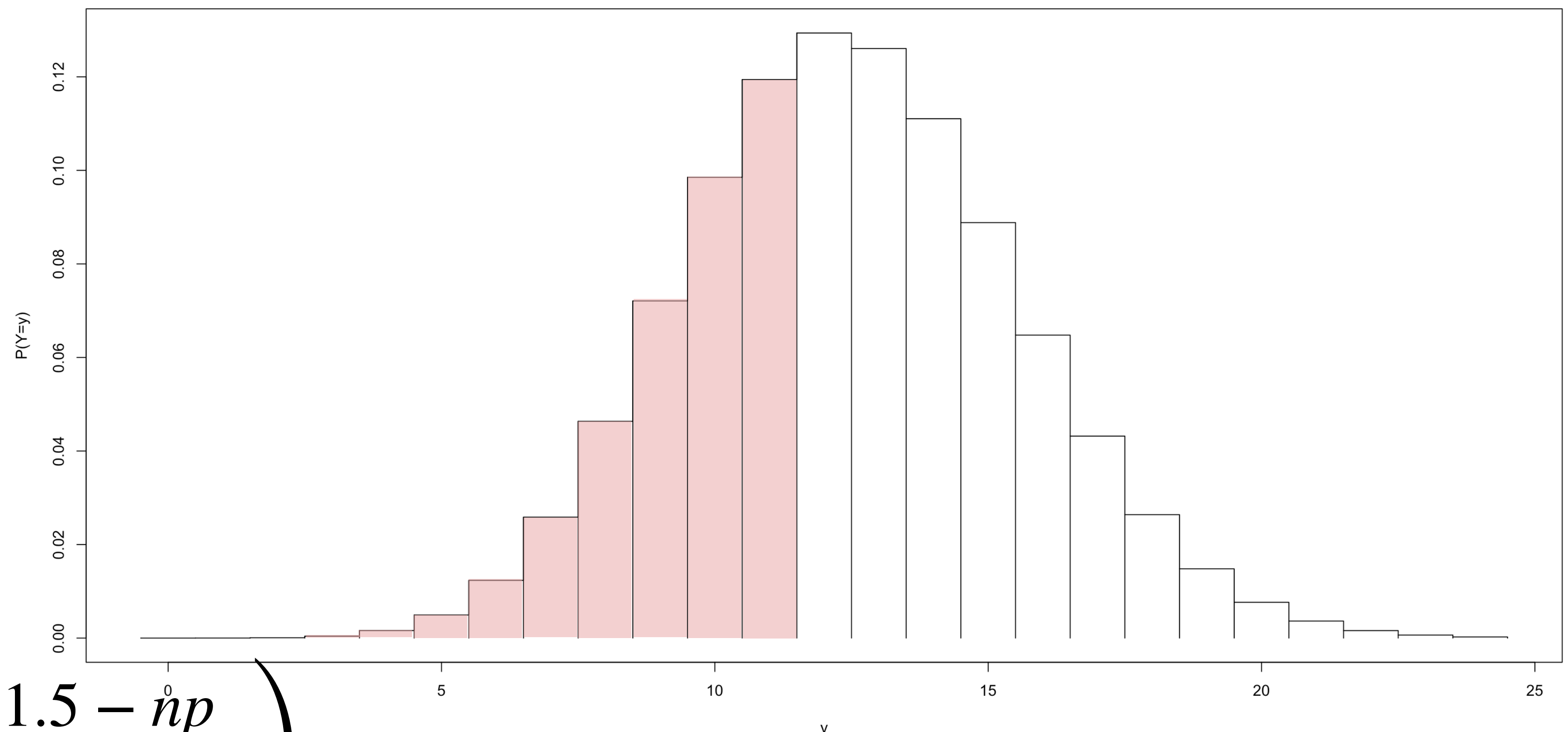
- La $Y = \sum_{i=1}^{n} X_i$. Hvilken fordeling har Y?

2. La $n = 50$ og $p = 0.25$, bestem så $P(Y \leq 11)$.

    - Merk: Vi har ikke tabell for disse verdiene av n og p. Hva kan vi gjøre i stedet?
      Hint: er n stor her?

Kontinuitetskorreksjon for binomisk fordeling

- Merk: Siden Y egentlig er en diskret stokastisk variabel, kan man få en mer presis approksimasjon ved å bruke kontinuitetskorreksjon når vi regner ut sannsynligheter.



$$
\begin{aligned}
P(Y \leq 11) &\approx P\left(Z \leq \frac{11.5 - np}{\sqrt{np(1-p)}}\right) \\
&\approx \Phi\left(\frac{11.5 - np}{\sqrt{np(1-p)}}\right) \\
&\approx \Phi\left(\frac{11.5 - 50 \cdot 0.25}{\sqrt{50 \cdot 0.25(1 - 0.25)}}\right) \\
&\approx \Phi(-0.33)
\end{aligned}
$$

Standard normalfordeling

$$\Phi(z) = P(Z \leq z)$$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.7	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
-3.6	.0002	.0002	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
-3.5	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002
-3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
-3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
-3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
-3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
-3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
-2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
-2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
-2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
-2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
-2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
-2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
-2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
-2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
-2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
-2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
-1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
-1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
-1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
-1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
-1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
-1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
-1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
-1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
-1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
-1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
-.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
-.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
-.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
-.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
-.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
-.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
-.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
-.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
-.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
-.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

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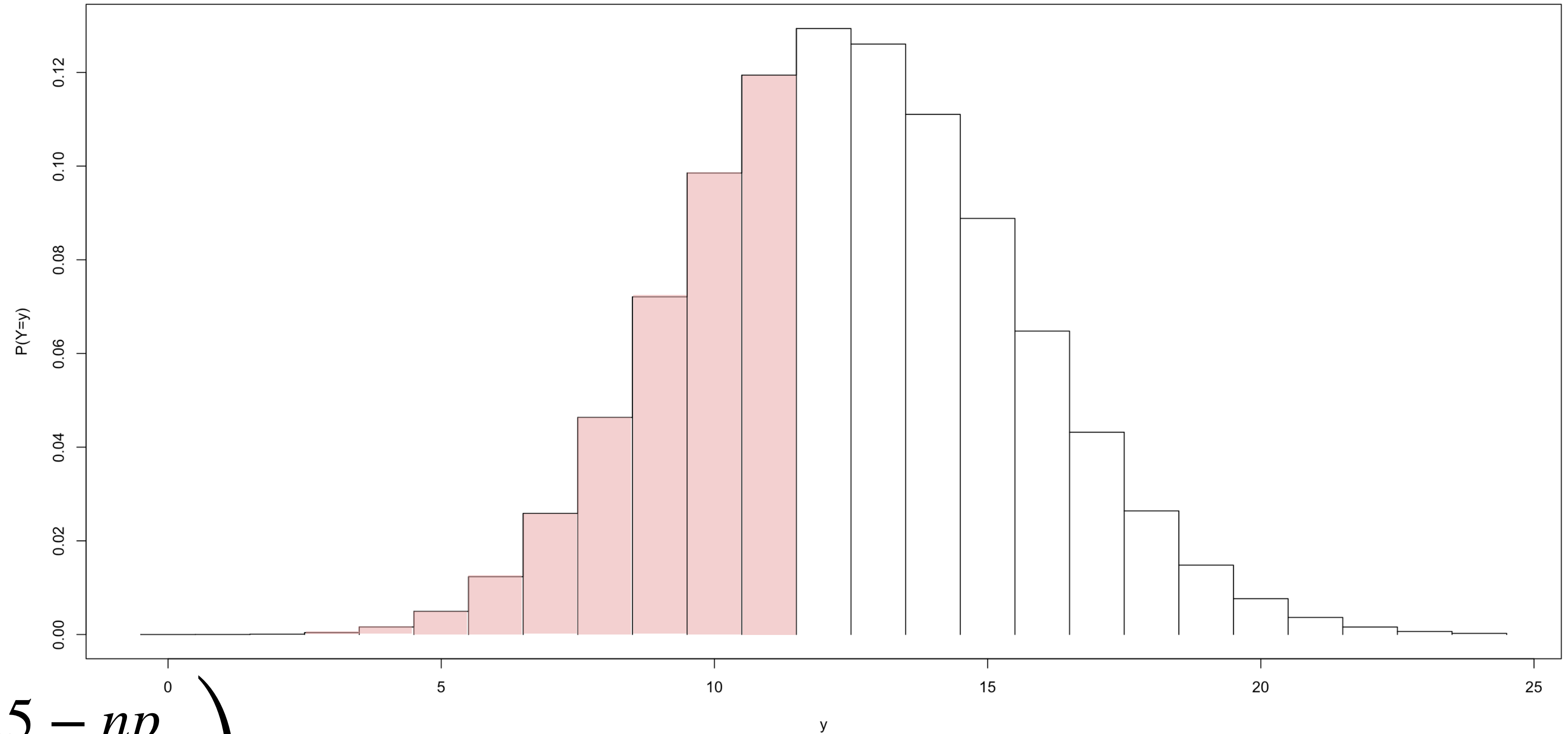

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$$\approx \Phi\left(\frac{11.5 - 50 \cdot 0.25}{\sqrt{50 \cdot 0.25(1 - 0.25)}}\right)$$

$$\approx \Phi(-0.33) = \underline{\underline{0.3707}}$$

Se temaside: "Regne ut sannsynligheter i en binomisk fordeling ved hjelp av normaltilnærming"