Plan for dagens undervisning

* Regne på eksempler/situasjoner som inkluderer
  - en regresjonsmodell som avviker litt fra vanlig enkel lineær regresjon
  - vurdering av en regresjonsmodell: residualplott
  - hvilken fordeling har $\widehat{\alpha}$, $\widehat{\sigma}^2$?
  - konfidensintervall for konstantleddet α

Regresjonsmodell regnet på forrige uke

- Anta $Y_1, Y_2, \ldots, Y_n$ uavhengige og $Y_i \sim N(\alpha + \beta x_i, \sigma^2 x_i^2)$
- Vi fant sannsynlighetsmaksimeringsestimatorene for α, β og σ^2 til å være

$$\widehat{\alpha} = \frac{\sum_{i=1}^n \frac{Y_i}{x_i^2} - \frac{1}{n}\left(\sum_{i=1}^n \frac{Y_i}{x_i}\right)\left(\sum_{i=1}^n \frac{1}{x_i}\right)}{\sum_{i=1}^n \frac{1}{x_i^2} - \frac{1}{n}\left(\sum_{i=1}^n \frac{1}{x_i}\right)^2}$$

$$\widehat{\beta} = \frac{1}{n}\sum_{i=1}^n \frac{Y_i}{x_i} - \widehat{\alpha} \cdot \frac{1}{n}\sum_{i=1}^n \frac{1}{x_i}$$

$$\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n \frac{(Y_i - \widehat{\alpha} - \widehat{\beta} x_i)^2}{x_i^2}$$

- Egenskaper for $\widehat{\alpha}$

$$E[\widehat{\alpha}] = \alpha, \qquad \text{Var}[\widehat{\alpha}] = \sigma^2 \cdot \frac{\sum_{i=1}^n \left(\frac{1}{x_i} - \frac{1}{n}\sum_{j=1}^n \frac{1}{x_j}\right)^2}{\left(\sum_{i=1}^n \frac{1}{x_i^2} - \frac{1}{n}\left(\sum_{i=1}^n \frac{1}{x_i}\right)^2\right)^2}$$

  - hvilken type fordeling har $\widehat{\alpha}$?
  - hvilken type fordeling har $\widehat{\sigma}^2$?

Hvilken fordeling har $\widehat{\alpha}$?

- Vi har at $\widehat{\alpha}$ er en lineær funksjon av Y_i'ene:

$$\widehat{\alpha} = \frac{\sum_{i=1}^{n} \frac{Y_i}{x_i^2} - \frac{1}{n}\left(\sum_{i=1}^{n} \frac{Y_i}{x_i}\right)\left(\sum_{i=1}^{n} \frac{1}{x_i}\right)}{\sum_{i=1}^{n} \frac{1}{x_i^2} - \frac{1}{n}\left(\sum_{i=1}^{n} \frac{1}{x_i}\right)^2}$$

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&= \frac{\sum_{i=1}^{n} \frac{Y_i}{x_i^2} - \frac{1}{n}\left(\sum_{i=1}^{n} \frac{Y_i}{x_i}\right)\left(\sum_{j=1}^{n} \frac{1}{x_j}\right)}{\sum_{j=1}^{n} \frac{1}{x_j^2} - \frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{x_j}\right)^2}
\end{aligned}
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&= \frac{\sum_{i=1}^{n} \frac{Y_i}{x_i^2} - \frac{1}{n}\left(\sum_{i=1}^{n} \frac{Y_i}{x_i}\right)\left(\sum_{j=1}^{n} \frac{1}{x_j}\right)}{\sum_{j=1}^{n} \frac{1}{x_j^2} - \frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{x_j}\right)^2} \\
&= \frac{\sum_{i=1}^{n} \frac{Y_i}{x_i^2} - \frac{1}{n}\sum_{i=1}^{n}\left[\frac{Y_i}{x_i}\sum_{j=1}^{n} \frac{1}{x_j}\right]}{\sum_{j=1}^{n} \frac{1}{x_j^2} - \frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{x_j}\right)^2}
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&= \frac{\sum_{i=1}^{n} \frac{Y_i}{x_i^2} - \frac{1}{n}\left(\sum_{i=1}^{n} \frac{Y_i}{x_i}\right)\left(\sum_{j=1}^{n} \frac{1}{x_j}\right)}{\sum_{j=1}^{n} \frac{1}{x_j^2} - \frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{x_j}\right)^2} \\
&= \frac{\sum_{i=1}^{n} \frac{Y_i}{x_i^2} - \frac{1}{n}\sum_{i=1}^{n}\left[\frac{Y_i}{x_i}\sum_{j=1}^{n} \frac{1}{x_j}\right]}{\sum_{j=1}^{n} \frac{1}{x_j^2} - \frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{x_j}\right)^2} \\
&= \frac{\sum_{i=1}^{n}\left[\frac{Y_i}{x_i^2} - \frac{1}{n}\frac{Y_i}{x_i}\sum_{j=1}^{n} \frac{1}{x_j}\right]}{\sum_{j=1}^{n} \frac{1}{x_j^2} - \frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{x_j}\right)^2}
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&= \frac{\sum_{i=1}^{n} \frac{Y_i}{x_i^2} - \frac{1}{n}\left(\sum_{i=1}^{n} \frac{Y_i}{x_i}\right)\left(\sum_{j=1}^{n} \frac{1}{x_j}\right)}{\sum_{j=1}^{n} \frac{1}{x_j^2} - \frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{x_j}\right)^2} \\
&= \frac{\sum_{i=1}^{n} \frac{Y_i}{x_i^2} - \frac{1}{n}\sum_{i=1}^{n}\left[\frac{Y_i}{x_i}\sum_{j=1}^{n} \frac{1}{x_j}\right]}{\sum_{j=1}^{n} \frac{1}{x_j^2} - \frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{x_j}\right)^2} \\
&= \frac{\sum_{i=1}^{n}\left[\frac{Y_i}{x_i^2} - \frac{1}{n}\frac{Y_i}{x_i}\sum_{j=1}^{n} \frac{1}{x_j}\right]}{\sum_{j=1}^{n} \frac{1}{x_j^2} - \frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{x_j}\right)^2} \\
&= \sum_{i=1}^{n}\left[\frac{\frac{Y_i}{x_i^2} - \frac{1}{n}\frac{Y_i}{x_i}\sum_{j=1}^{n} \frac{1}{x_j}}{\sum_{j=1}^{n} \frac{1}{x_j^2} - \frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{x_j}\right)^2}\right]
\end{aligned}
$$

Hvilken fordeling har $\widehat{\alpha}$?

- Vi har at $\widehat{\alpha}$ er en lineær funksjon av Y_i'ene:

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\widehat{\alpha} &= \frac{\sum_{i=1}^{n} \frac{Y_i}{x_i^2} - \frac{1}{n}\left(\sum_{i=1}^{n} \frac{Y_i}{x_i}\right)\left(\sum_{i=1}^{n} \frac{1}{x_i}\right)}{\sum_{i=1}^{n} \frac{1}{x_i^2} - \frac{1}{n}\left(\sum_{i=1}^{n} \frac{1}{x_i}\right)^2} \\
&= \frac{\sum_{i=1}^{n} \frac{Y_i}{x_i^2} - \frac{1}{n}\left(\sum_{i=1}^{n} \frac{Y_i}{x_i}\right)\left(\sum_{j=1}^{n} \frac{1}{x_j}\right)}{\sum_{j=1}^{n} \frac{1}{x_j^2} - \frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{x_j}\right)^2} \\
&= \frac{\sum_{i=1}^{n} \frac{Y_i}{x_i^2} - \frac{1}{n}\sum_{i=1}^{n}\left[\frac{Y_i}{x_i}\sum_{j=1}^{n} \frac{1}{x_j}\right]}{\sum_{j=1}^{n} \frac{1}{x_j^2} - \frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{x_j}\right)^2} \\
&= \frac{\sum_{i=1}^{n}\left[\frac{Y_i}{x_i^2} - \frac{1}{n}\frac{Y_i}{x_i}\sum_{j=1}^{n} \frac{1}{x_j}\right]}{\sum_{j=1}^{n} \frac{1}{x_j^2} - \frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{x_j}\right)^2} \\
&= \sum_{i=1}^{n}\left[\frac{\frac{Y_i}{x_i^2} - \frac{1}{n}\frac{Y_i}{x_i}\sum_{j=1}^{n} \frac{1}{x_j}}{\sum_{j=1}^{n} \frac{1}{x_j^2} - \frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{x_j}\right)^2}\right] \\
&= \sum_{i=1}^{n}\left[Y_i \cdot \frac{\frac{1}{x_i^2} - \frac{1}{n}\frac{1}{x_i}\sum_{j=1}^{n} \frac{1}{x_j}}{\sum_{j=1}^{n} \frac{1}{x_j^2} - \frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{x_j}\right)^2}\right]
\end{aligned}$$

Hvilken fordeling har $\widehat{\alpha}$?

⋆ Vi har at $\widehat{\alpha}$ er en lineær funksjon av Y_i'ene:

$$
\begin{aligned}
\widehat{\alpha} &= \frac{\sum_{i=1}^n \frac{Y_i}{x_i^2} - \frac{1}{n}\left(\sum_{i=1}^n \frac{Y_i}{x_i}\right)\left(\sum_{i=1}^n \frac{1}{x_i}\right)}{\sum_{i=1}^n \frac{1}{x_i^2} - \frac{1}{n}\left(\sum_{i=1}^n \frac{1}{x_i}\right)^2} \\
&= \frac{\sum_{i=1}^n \frac{Y_i}{x_i^2} - \frac{1}{n}\left(\sum_{i=1}^n \frac{Y_i}{x_i}\right)\left(\sum_{j=1}^n \frac{1}{x_j}\right)}{\sum_{j=1}^n \frac{1}{x_j^2} - \frac{1}{n}\left(\sum_{j=1}^n \frac{1}{x_j}\right)^2} \\
&= \frac{\sum_{i=1}^n \frac{Y_i}{x_i^2} - \frac{1}{n}\sum_{i=1}^n \left[\frac{Y_i}{x_i}\sum_{j=1}^n \frac{1}{x_j}\right]}{\sum_{j=1}^n \frac{1}{x_j^2} - \frac{1}{n}\left(\sum_{j=1}^n \frac{1}{x_j}\right)^2} \\
&= \frac{\sum_{i=1}^n \left[\frac{Y_i}{x_i^2} - \frac{1}{n}\frac{Y_i}{x_i}\sum_{j=1}^n \frac{1}{x_j}\right]}{\sum_{j=1}^n \frac{1}{x_j^2} - \frac{1}{n}\left(\sum_{j=1}^n \frac{1}{x_j}\right)^2} \\
&= \sum_{i=1}^n \left[\frac{\frac{Y_i}{x_i^2} - \frac{1}{n}\frac{Y_i}{x_i}\sum_{j=1}^n \frac{1}{x_j}}{\sum_{j=1}^n \frac{1}{x_j^2} - \frac{1}{n}\left(\sum_{j=1}^n \frac{1}{x_j}\right)^2}\right] \\
&= \sum_{i=1}^n \left[Y_i \cdot \frac{\frac{1}{x_i^2} - \frac{1}{n}\frac{1}{x_i}\sum_{j=1}^n \frac{1}{x_j}}{\sum_{j=1}^n \frac{1}{x_j^2} - \frac{1}{n}\left(\sum_{j=1}^n \frac{1}{x_j}\right)^2}\right] = \sum_{i=1}^n a_i Y_i
\end{aligned}
$$

Student t-fordeling

Definisjon (t-fordeling)

La Z og V være uavhengige stokastiske variabler, og la $Z \sim N(0,1)$ og $V \sim \chi^2_\nu$. La så T være en stokastisk variabel definert ved

$$T = \frac{Z}{\sqrt{V/\nu}}.$$

Da sies T å være (Student) t-fordelt med ν frihetsgrader.