Problem 1 Continuous fields

Consider a stationary Gaussian random field with mean 0 and variance 1.

For distance $\tau = ||\mathbf{x} - \mathbf{x}'||$ between spatial locations $\mathbf{x}$ and $\mathbf{x}'$, two valid correlation functions are $\rho_A(\tau) = \exp(-1.5\tau)$ and $\rho_B(\tau) = \exp(-0.75\tau^2)$.

a) Sketch the correlation functions with distance τ on the first axis.

Compute the effective correlation range, i.e. the distance at which the correlation gets below 0.05 for both functions.

Figure 1 shows one realization from each of the two correlation functions on $[0, 10] \subset \mathcal{R}$. Which display corresponds to ρ_A and ρ_B?

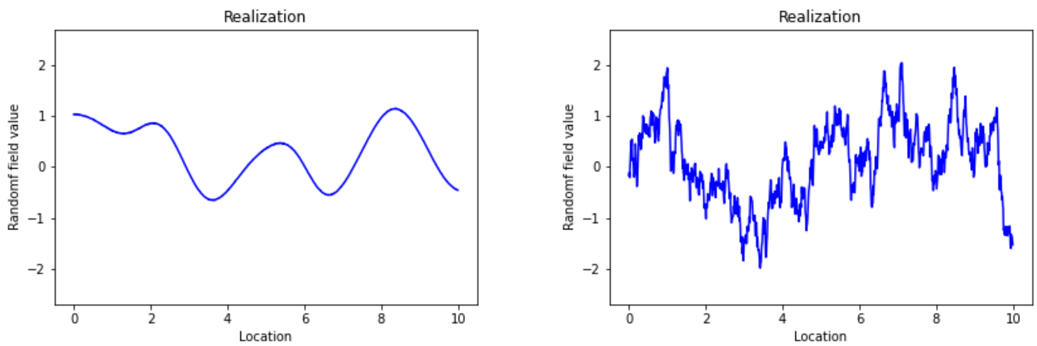

Figure 1: One realization based on correlation function ρ_A and one based on ρ_B. Which of the displays is A and B?

Consider next a two dimensional random field $\{r(\mathbf{x}), \mathbf{x} \in \mathcal{D} \subset \mathcal{R}^2\}$. Assume we know that $r(0, 0) = -2$ and $r(1, 1) = 0.5$.

b) Derive the conditional mean for $r(1.1, 1.1)$ for both correlation functions ρ_A and ρ_B.

Comment on the differences.

Hint: Recall that the inverse of a 2×2 matrix is

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}^{-1} = \frac{1}{a^2 - b^2} \begin{bmatrix} a & -b \\ -b & a \end{bmatrix}$$

.

c) Could you simplify the solution in (b) for the case with ρ_A?

From now on we use correlation function ρ_A to model the random field $\{r(\mathbf{x}), \mathbf{x} \in \mathcal{D} \subset \mathcal{R}^2\}$. Assume that one observes the random field at the locations $(0,0)$ and $(1,1)$ with noise. The observation model is $p(d(\mathbf{x}^o)|r(\mathbf{x}^o)) = \phi(d(\mathbf{x}^o); r(\mathbf{x}^o), 1^2)$ for $\mathbf{x}^o = (0,0)$ and $\mathbf{x}^o = (1,1)$, assuming conditional independence. Here, $\phi(d; \mu, \sigma^2)$ is the Gaussian density function with mean μ and variance σ^2 evaluated at d . The measurements are $d(0,0) = -2$ and $d(1,1) = 0.5$.

d) Use prior and likelihood models to show that the conditional mean for $r(1.1, 1.1)$ is given by:

$$\hat{r} = \begin{bmatrix} 0.097 \\ 0.809 \end{bmatrix}^t \begin{bmatrix} 0.502 & -0.030 \\ -0.030 & 0.502 \end{bmatrix} \begin{bmatrix} -2 \\ 0.5 \end{bmatrix} = -2 \cdot 0.024 + 0.5 \cdot 0.403 = 0.153$$

Comment on the differences with that from (b).

What is the associated prediction variance?

e) Consider a point $\mathbf{x}$ in the vicinity of $(1,1)$ and such that $||\mathbf{x} - (0,0)|| = \sqrt{2}$.

For what $\mathbf{x}$ do the conditional mean for $r(\mathbf{x})$ exceed 0?

Hint: Recall that for a circle with radius R, the length of a chord (line segment) spanning angle θ is $2R\sin\left(\frac{\theta}{2}\right)$.

Problem 2 Event random fields

Let $\{k(\mathbf{x}); \mathbf{x} \in \mathcal{D} \subset \mathcal{R}^2\}$ be an event random field.

a) List assumptions required for $\{k(\mathbf{x}); \mathbf{x} \in \mathcal{D} \subset \mathcal{R}\}$ to be a Poisson random field.

What is the difference between a homogeneous and nonhomogeneous Poisson random field?

b) Figure 2 Left) shows event location on the unit square and Figure 2 Right) shows the number of event in disjoint cells on a regular 3×3 grid covering the domain. Given the total number of events k, what is the marginal distribution for the number of events in one cell?

One hypothesis test of the Poisson random field assumption is based on the statistic $G = \sum_{i=1}^{C} (O_i - E_i)^2 / E_i$, where O_i is the observed number of events in cell i, E_i is the expected number of events in cell i, and C is the number of cells. Under the null hypothesis of a Poisson model, the test statistic is approximately χ^2_{C-2} distributed when the Poisson intensity parameter is estimated.

Use the test statistic to study whether the realization and numbers in Figure 2 are representing data from a homogeneous Poisson random field model.

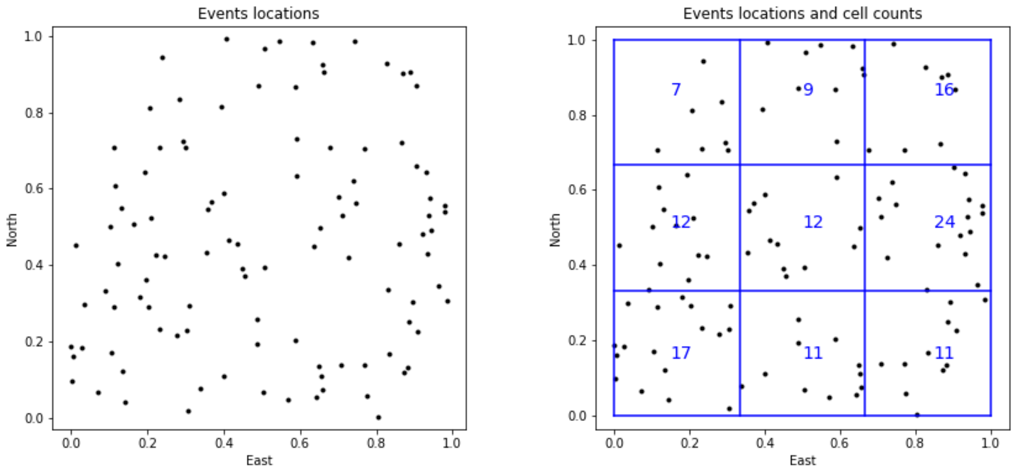

Figure 2: Left) Events locations on the unit square. Right) Events and counts in 9 cells.

c) Considering a homogeneous Poisson random field and selecting position $\mathbf{x}_0 = (0.5, 0.5)$, derive the density function for the distance to the nearest event point from $\mathbf{x}_0$?

Let the distance to the nearest event point from $\mathbf{x}_0$ be 0.05. Conditional on this, derive the density function to the second nearest event point from $\mathbf{x}_0$.

Problem 3 Mosaic fields

Consider a Markov random field $\{l(\mathbf{x}); \mathbf{x} \in \mathcal{D} \subset \mathcal{R}^2\}$ with $l(\mathbf{x}) \in \{0, 1\}$, where $\mathcal{D}$ is a grid of $n_1 \times n_2$ cells.

a) Define the joint probability and the full conditional formulation for the Ising model.

Write an algorithm (using pseudo-code, no specific syntax required) for generating a realization from the Ising model using a version of Markov chain Monte Carlo sampling.

b) Consider now a second-order neighborhood for the Markov random field.

Define the maximal clique configurations for this situation.

If certain properties of the field are wanted, what symmetry assumptions are reasonable for the clique potentials? Discuss, and use sketches to clarify your answer.

Modify your algorithm in (a) for generating a realization from this second-order neighborhood Markov random field model using Markov chain Monte Carlo sampling.