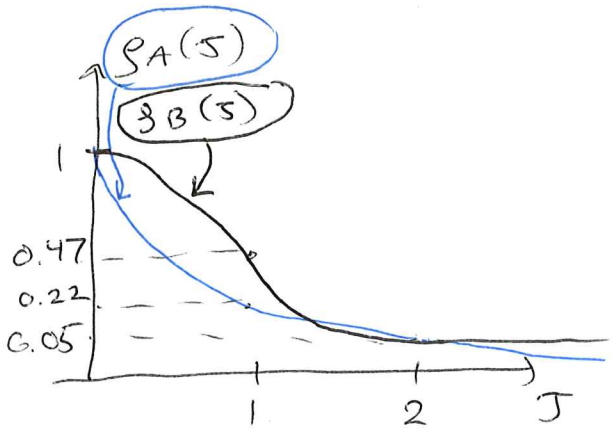
TMA4250, 2 June '25

1.

a)



$\tau = 1$: $e^{-1.5} = 0.22$

$e^{-0.75} = 0.47$

Effective correlation range $e^{-3} = 0.05$.

$\rho^A(\tau) = 0.05 = e^{-1.5\cdot\tau} \longleftrightarrow \tau = \frac{\ln 0.05}{-1.5} = \underline{2}$

$\rho^B(\tau) = 0.05 = e^{-0.75\tau^2} \longleftrightarrow \tau = \sqrt{\frac{\ln 0.05}{-0.75}} = \underline{2}$

ρ_A: The exponential correlation function has faster decline of the function near $\tau = 0$.

ρ_B: The Gaussian correlation function is smoother.

The ~~left~~ display is ρ_B realization. [Left]

The ~~right~~ display is ρ_A realization. [Right]

i)

1b)

$$E\left(r(1,1) \mid r(0,0), r(1,1)\right) = \hat{r} = \sum_{[(1,1),(0,0)],[(1,1),(1,1)]} \sum^{-1}_{\substack{(0,0)\\(1,1)}} \cdot \begin{pmatrix} d(0,0) \\ d(1,1) \end{pmatrix}$$

pA:

$$\sum^{-1}_{\substack{(0,0)\\(1,1)}} = \begin{bmatrix} 1 & e^{-1.5\cdot\sqrt{2}} \\ e^{-1.5\sqrt{2}} & 1 \end{bmatrix}^{-1} = \begin{pmatrix} \frac{1}{1-e^{-1.5\cdot 2\sqrt{2}}} & -\frac{e^{-1.5\sqrt{2}}}{1-e^{-1.5\cdot 2\cdot\sqrt{2}}} \\ -\frac{e^{-1.5\sqrt{2}}}{1-e^{-1.5\cdot 2\cdot\sqrt{2}}} & \frac{1}{1-e^{-1.5\cdot 2\cdot\sqrt{2}}} \end{pmatrix}$$

$$= \begin{pmatrix} 1.015 & -0.122 \\ -0.122 & 1.015 \end{pmatrix}$$

$$\sum_{[(1,1),(0,0)],[(1,1),(1,1)]} = \left(e^{-1.5\cdot 1.56}, e^{-1.5\cdot 0.14}\right) = (0.097, 0.809)$$

$$\hat{r} = (0.097, 0.809) \cdot \begin{bmatrix} 1.015\cdot(-2) - 0.122(0.5) \\ -0.122(-2) + 1.015(0.5) \end{bmatrix}$$

$$= \underline{0.405}$$

The exponential correlation has a Markov property, so

$$E(r(1,1,1) \mid r(\blacksquare,\blacksquare)) = e^{-1.5\cdot 0.14} \cdot \frac{1}{1} \cdot 0.5 = \underline{0.405}$$

It is enough to know $r(1,1)$. $[0.097\cdot 1.015 - 0.809\cdot 0.122 = 0]$

ii

1b)

$$\Sigma^{-1}_{\substack{(0,0),\\(1,1)}} = \begin{pmatrix} \frac{1}{1-e^{-0.75\cdot 2\cdot 2}} & -\frac{e^{-0.75\cdot 2}}{1-e^{-0.75\cdot 2\cdot 2}} \\ -\frac{e^{-0.75\cdot 2}}{1-e^{-0.75\cdot 2\cdot 2}} & \frac{1}{1-e^{-0.75\cdot 2\cdot 2}} \end{pmatrix}$$

$$= \begin{pmatrix} 1.052 & -0.235 \\ -0.235 & 1.052 \end{pmatrix}$$

$$\Sigma_{((1,1),(0,0)),((1,1),(1,1)),(1,1))} = \left(e^{-0.75\cdot 2.42}, e^{-0.75\cdot 0.02}\right) = (0.163, 0.985)$$

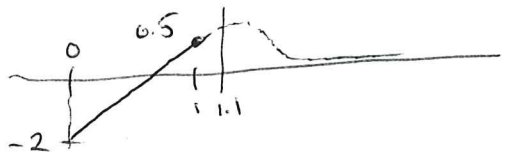
$$\hat{r} = (0.163, 0.985) \begin{bmatrix} 1.052\cdot(-2) - 0.235(0.5) \\ -0.235(-2) + 1.052(0.5 \end{bmatrix}$$

$$= \underline{0.62}$$

1c)

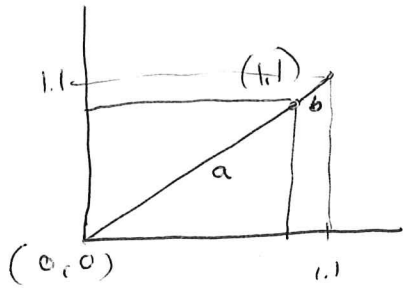
There is no Markov property here.

In fact the prediction is larger than 0.5, because it inherits some of the slope from the -2 measurement at (0,0)



iii)

10)



$$c = \|(1.1,1.1) - (0,0)\|$$
$$= \|(1,1) - (0,0)\| + \|(1.1,1.1) - (1,1)\|$$
$$= a + b$$

$$a = \sqrt{2}, \quad b = \sqrt{2 \cdot 0.1^2} = \underline{\sqrt{2} \cdot 0.1}$$

$$c = \sqrt{2 \cdot 1.1^2} = \underline{\sqrt{2} \cdot 1.1}$$

Then $\left(e^{-1.5\sqrt{2}\,1.1}, e^{-1.5\sqrt{2}\,0.1} \right) \begin{pmatrix} 1 \\ -e^{-1.5\sqrt{2}} \end{pmatrix}$

$$= e^{-1.5\sqrt{2}\,1.1} - e^{-1.5\sqrt{2}(0.1+1)} = \underline{0}.$$

There is 0 weight on $d(0,0)$ in $\hat{r}$.

The Markov property means that it is sufficient to know $r(1,1)$.

(iv)

1d) Prior

$$\underline{r} = \begin{pmatrix} r(1.1,1.1) \\ r(0,0) \\ r(1,1) \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \underbrace{\begin{pmatrix} 1 & e^{-1.5\cdot 1.56} & e^{-1.5\cdot 0.14} \\ & 1 & e^{-1.5\sqrt{2}} \\ & & 1 \end{pmatrix}}_{\Sigma} \right)$$

Likelihood:

$$\underline{d}|\underline{r} \sim N\left(\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{H} \cdot \underline{r}, \begin{pmatrix} 1^2 & 0 \\ 0 & 1^2 \end{pmatrix} \right)$$

Posterior:

$$r|d \sim N\left(\Sigma H^T \left[H\Sigma H^T + \Sigma_{d|r} \right]^{-1} \cdot \underline{d}, \Sigma - \Sigma H^T \left[H\Sigma H^T + \Sigma_{d|r} \right]^{-1} H\Sigma \right)$$

We are only interested in $r(1.1,1.1)$,

$$\hat{r} = \left(e^{-1.5\cdot 1.56}, e^{-1.5\cdot 0.14} \right) \begin{bmatrix} 2 & e^{-1.5\sqrt{2}} \\ e^{-1.5\sqrt{2}} & 2 \end{bmatrix}^{-1} \cdot \begin{pmatrix} d(0,0) \\ d(1,1) \end{pmatrix}$$

$$= (0.097, 0.809) \cdot \begin{bmatrix} \frac{2}{4 - e^{-1.5\cdot\sqrt{2}\cdot 2}} & \frac{-e^{-1.5\sqrt{2}}}{4 - e^{-1.5\sqrt{2}\cdot 2}} \\ \frac{-e^{-1.5\sqrt{2}}}{4 - e^{-1.5\sqrt{2}\cdot 2}} & \frac{2}{4 - e^{-1.5\sqrt{2}\cdot 2}} \end{bmatrix} \begin{pmatrix} -2 \\ 0.5 \end{pmatrix}$$

$$= (0.097, 0.809) \begin{bmatrix} 0.502 & -0.03 \\ -0.03 & 0.502 \end{bmatrix} \begin{pmatrix} -2 \\ 0.5 \end{pmatrix}$$

$$= \underline{0.153}$$

Compared with 1b) it is closer to 0.
Less influenced by data, more by prior.

(d) Prediction variance:

$$\text{Var}\left(r(1.1,1.1) \mid d(0,0), d(1,1)\right)$$

$$= 1 - (0.024, 0.403)\begin{pmatrix} 0.097 \\ 0.809 \end{pmatrix}$$

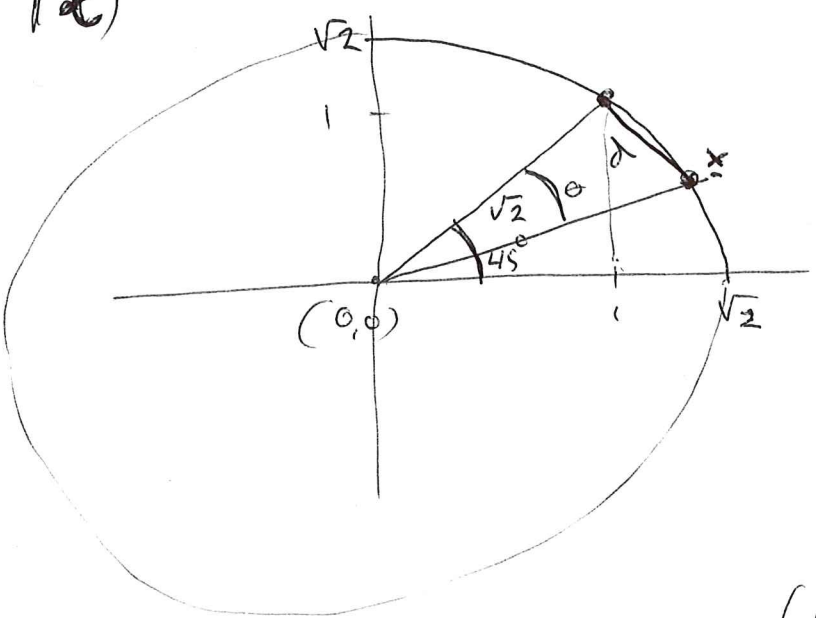
$$= 1 - 0.33 = \underline{0.67}.$$

Formula comes from

$$\text{Var}(\underline{r} \mid \underline{d}) = \Sigma_r - \Sigma H^T \left[H \Sigma H^T + \Sigma_{d|r} \right]^{-1} H \Sigma$$

We are only interested in $r(1.1, 1.1)$ variance.

(e)



Distance from $(0,0)$ to $\underline{x}$ is $\sqrt{2}$.

Let d be distance from $(1,1)$ to $\underline{x}$

$$\hat{r} = \left[e^{-1.5 \cdot \sqrt{2}}, e^{-1.5 \cdot d} \right] \cdot \begin{pmatrix} 0.502 & -0.03 \\ -0.03 & 0.502 \end{pmatrix} \begin{pmatrix} -2 \\ 0.5 \end{pmatrix}$$

$$= e^{-1.5\sqrt{2}} \cdot \left(0.502(-2) - 0.03(0.5)\right) + e^{-1.5 \cdot d} \left(-0.03(-2) + 0.502 \cdot 0.5\right)$$

$$= -0.122 + 0.31 \cdot e^{-1.5 \cdot d}$$

vi)

e)

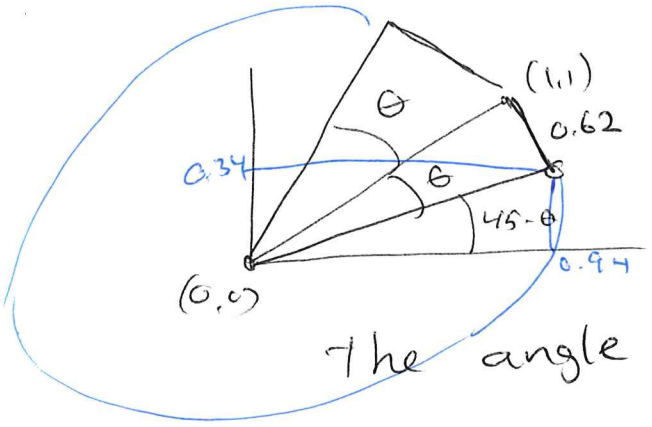
For $d=0$, $\hat{r} = -0.122 + 0.31 = \underline{0.189}$.

For large d, $\hat{r}$ gets negative.

Solve for $\hat{r} = 0$:

$$-1.5 \cdot d = \ln\left(\frac{0.122}{0.31}\right) = -0.93$$

$$d = \frac{-0.93}{-1.5} = \underline{0.62}.$$



The angle is obtained by:

$$0.62 = 2\sqrt{2} \cdot \sin\left(\frac{\theta}{2}\right)$$

$$\theta = 2 \cdot \arcsin\left(\frac{0.62}{2\sqrt{2}}\right) = 0.44 = \underline{25^{\circ}}.$$

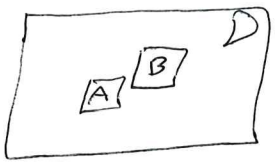
$45 - \theta = \underline{26}$.

$$\underline{x} = \begin{pmatrix} \cos(20) \\ \sin(20) \end{pmatrix} = \underline{\begin{pmatrix} 0.94 \\ 0.34 \end{pmatrix}}.$$

vii)

2.

a)



k_A = number of events in $A \subset D$

k_B = number of events in $B \subset D$, $A \cap B = \phi$

* $\begin{cases} k_A \sim \text{Poisson}(\lambda_A) \\ k_B \sim \text{Poisson}(\lambda_B) \end{cases}$
* k_A and k_B are independent when $A \cap B = \emptyset$

A homogeneous Poisson process has constant intensity parameter λ, such that for a subdomain $A \subset D$, $\lambda_A = \int_{x \in A} \lambda \, dx = \lambda |A|$.

A non-homogeneous Poisson process has spatially varying intensity $\lambda(x)$, such that

$$\lambda_A = \int_{x \in A} \lambda(x) dx .$$

b)

$$k_A | k \sim \text{Binomial}\left(k, \frac{\lambda_A}{\lambda_D}\right)$$

k: number of trials; $\frac{\lambda_A}{\lambda_D}$: success probability.

For a homogeneous Poisson process $\frac{\lambda_A}{\lambda_D} = \frac{1}{9}$.

Here, $k = 7+9+16+12+12+24+17+11+11 = 119$

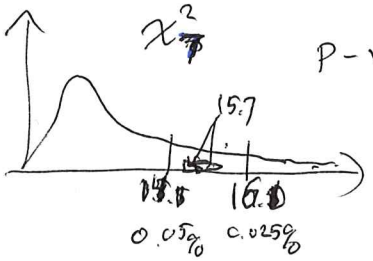
viii)

2b) The test statistic becomes

$$G = \sum_{i=1}^{9} \frac{(O_i - 13.2)^2}{13.2}$$

$$= 2.93 + 1.35 + 0.58 + 0.11 + 0.11 + 8.78 + 1.08 + 0.37 + 0.37$$

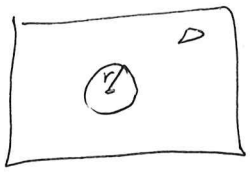
$$= \underline{15.7}$$



P-value ≈ 0.03

This is quite small, but not extremely small (could be random variation).

c)



R_1 = distance to nearest event location

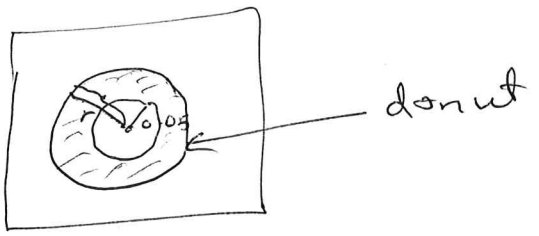
$P(R_1 > r) = P(0 \text{ events in circle with radius } r)$

$$\overset{\text{Poisson}}{=} \frac{(\lambda \pi r^2)^0}{0!} \cdot e^{-\lambda \pi r^2} = e^{-\lambda \pi r^2}$$

Density $f_{R_1}(r) = \frac{\partial}{\partial r}\left[1 - P(R_1 > r)\right] = \underline{2\lambda \pi r e^{-\lambda \pi r^2}}$

This is a Weibull distribution with shape parameter 2 and scale parameter $1/\sqrt{\lambda \pi}$.

c) R_2 = distance to second nearest event location



$$P(R_2 > r \mid R_1 = 0.05) = P(0 \text{ events in donut})$$

$$= e^{-\lambda\pi(r^2 - 0.05^2)}$$

Density

$$f_{R_2}(r) = 2\lambda\pi r \cdot e^{-\lambda\pi(r^2 - 0.05^2)} \qquad r > 0.05.$$

3a)

Joint formulation

$$p(\ell) = \text{const.} \prod_c V_c(\ell_j ; j \in c)$$

$$= \text{const.} \exp\left\{ \beta \sum_{i \sim j} I(\ell_i = \ell_j) \right\}$$

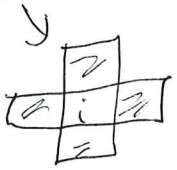
$i \sim j$ cliques are vertical or horizontal nearest neighbors on grid

Full conditional:

$$p(\ell_i \mid \ell_j ; j \neq i) \underset{\text{Markov}}{=} p(\ell_i \mid \ell_j ; j \in N_i)$$

$$\propto \exp\left(\beta \sum_{j \in N_i} I(\ell_i = \ell_j) \right)$$

N_i: four nearest neighbors



N_i is neighborhood of cell i.

x)

Algorithms,

A) Gibbs sampling

Initiate

$\underline{l}^0 = (l_1^0, \ldots l_n^0)$, $n = n_1 n_2$

Iterate $k = 0, 1, 2, \ldots,$ Iter

- draw random cell i
- sample new l_i^* from $p(l_i \mid l_j^k ; j \in N_i)$

  set $\underline{l}^{k+1} = \underline{l}^k$, $l_i^{k+1} = l_i^*$.

Ising full conditional prob.

B) Metropolis-Hastings

Initiate

$\underline{l}^0 = (l_1^0, \ldots l_n^0)$, $n = n_1 n_2$

Iterate $k = 0, 1, 2, \ldots,$ Iter

- draw random cell i
- set $l_i^* = \begin{cases} 0 & \text{if } l_i^k = 1 \\ 1 & \text{if } l_i^k = 0 \end{cases}$
- evaluate acceptance rate

$$\alpha = \min\left\{1, \frac{p(l_i^* \mid l_j ; j \in N_i)}{p(l_i^k \mid l_j ; j \in N_i)}\right\}$$

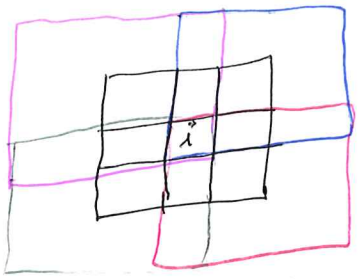
- $\underline{l}^{k+1} = \underline{l}^k$
- if $U(0,1) > \alpha$

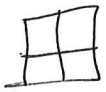
  set $l_i^{k+1} = l_i^*$

There are other variant of these algorithms.

When Iter $\to \infty$, they produce exact (dependent) samples from $p(\underline{l})$.

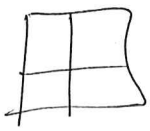
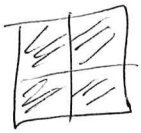
3b) Second order neighborhood



i is in all these neighborhoods, maximal cliques are  , 2×2 cells.

Model $p(\underline{l}) \propto \exp\left(\sum_c \Phi_c(l_c)\right)$

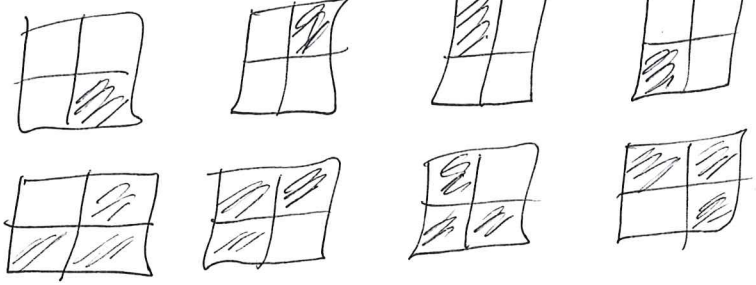
If symmetry is wanted

2 comb.   ← Φ_{equal} potential

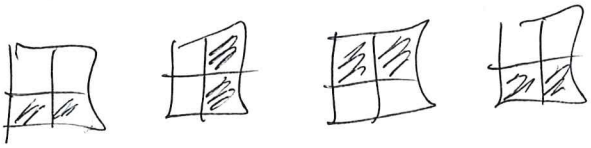
all white all black

should have same clique potential Φ_c

Corners should be the same

8 comb.  Φ_{corner} potential

Half should be same vertical / horizontal

4 comb.  Φ_{half} potential

Diagonals

2 comb.  Φ_{diag} potential

The Gibbs or Metropolis-Hastings sampling is similar (using single-site updating in each iteration).

The difference is in the full conditional probabilities which will now be $\{q_{equal}, q_{corner}, q_{half}, q_{diag}\}$ rather than just B or O.