Problem 1 Continuous fields

A common correlation function is the square exponential, also known as the Gaussian. Assume a prior stationary Gaussian random field $\{r(\mathbf{x}), \mathbf{x} \in (0,1) \times (0,1)\}$ has mean 0, marginal variance 1 and a Gaussian correlation function.

a) Sketch and describe the Gaussian correlation functions for two selections of parameter values of your choice.

For your parameterization of the Gaussian correlation function, find the distance $\tau = ||\mathbf{x} - \mathbf{x}'||$ between two locations $\mathbf{x}$ and $\mathbf{x}'$ at which the correlation equals 0.5.

For your parameterization of the Gaussian correlation function, find the distance at which the correlation function has second derivative with respect to location equal to 0.

At a location $\mathbf{x}_0 \in \mathcal{D}$ one makes an observation $d_0 = 1$. The observation model (likelihood model) is $d_0 = r(\mathbf{x}_0) + \epsilon_0$, for independent measurement error $\epsilon_0 \sim N(0, 0.1^2)$.

b) Derive the posterior distribution of $r(\mathbf{x}_0)$ given $d_0 = 1$.

c) At what distance $\tau = ||\mathbf{x} - \mathbf{x}_0||$ away from the data location will the posterior standard deviation be half of that in the prior model?

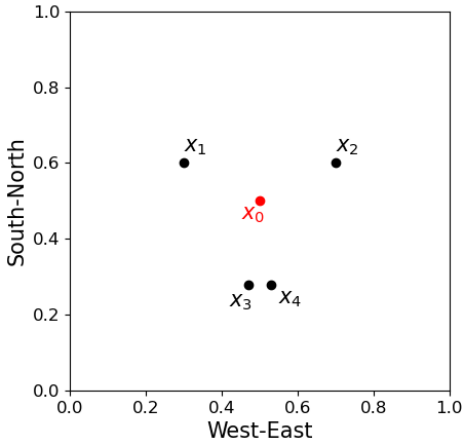


Figure 1: Illustration of four data locations (black) and a single prediction location (red).

Figure 1 shows a situation with four observation locations; $\mathbf{x}_1 = (0.3, 0.6)'$, $\mathbf{x}_2 = (0.7, 0.6)'$, $\mathbf{x}_3 = (0.47, 0.26)'$, $\mathbf{x}_4 = (0.53, 0.26)'$ where observations are d_1, d_2, d_3 and d_4, respectively, along with a single prediction location $\mathbf{x}_0 = (0.5, 0.5)'$. The data is measured with model $d_i = r(\mathbf{x}_i) + \epsilon_i$, $i = 1, 2, 3, 4$ for independent zero-mean Gaussian noise terms ϵ_i having variance $\sigma^2_{d|r} = 0.1^2$.

d) Show that the posterior mean of $r(\mathbf{x}_0)$ at the prediction location, given the observations, can be written $\hat{r}(\mathbf{x}_0) = w_1 d_1 + w_2 d_2 + w_3 d_3 + w_4 d_4$, for weights w_1, w_2, w_3, w_4. Express weights $\mathbf{w} = (w_1, w_2, w_3, w_4)'$ using covariance matrix notation.

We assume a Gaussian correlation function with effective spatial correlation range 0.3 for the Gaussian random field $\{r(\mathbf{x}), \mathbf{x} \in (0, 1) \times (0, 1)\}$. Which one of the following weights combinations is correct?

1. $w_1 = 0.25$, $w_2 = 0.25$, $w_3 = 0.25$, $w_4 = 0.25$.
2. $w_1 = 0.18$, $w_2 = 0.18$, $w_3 = 0.1$, $w_4 = 0.1$.
3. $w_1 = 0.34$, $w_2 = 0.34$, $w_3 = 0.16$, $w_4 = 0.16$.

Discuss each case and provide arguments for your choice, without going to mathematical detail.

Problem 2 Event random fields

a) Figure 2 shows single realizations from three different spatial point processes. Suggest the underlying models and parameter settings and elaborate for your answer.

b) Consider a non-homogeneous Poisson random field model on the unit square $D = [0, 1] \times [0, 1]$ with intensity function $\lambda(\mathbf{x}) = 20 x_1 x_2$, $x_1 \in [0, 1]$, $x_2 \in [0, 1]$. Let $k = k_D$ denote the number of points in the entire domain.

Calculate the expected value of k.

Calculate the probability that $k \leq 2$.

Define further $A = [0.5, 1] \times [0.5, 1]$, and let k_A denote the number of points in sub-domain A.

Calculate the probability that $k = 5$ given that $k_{A_4} = 5$?

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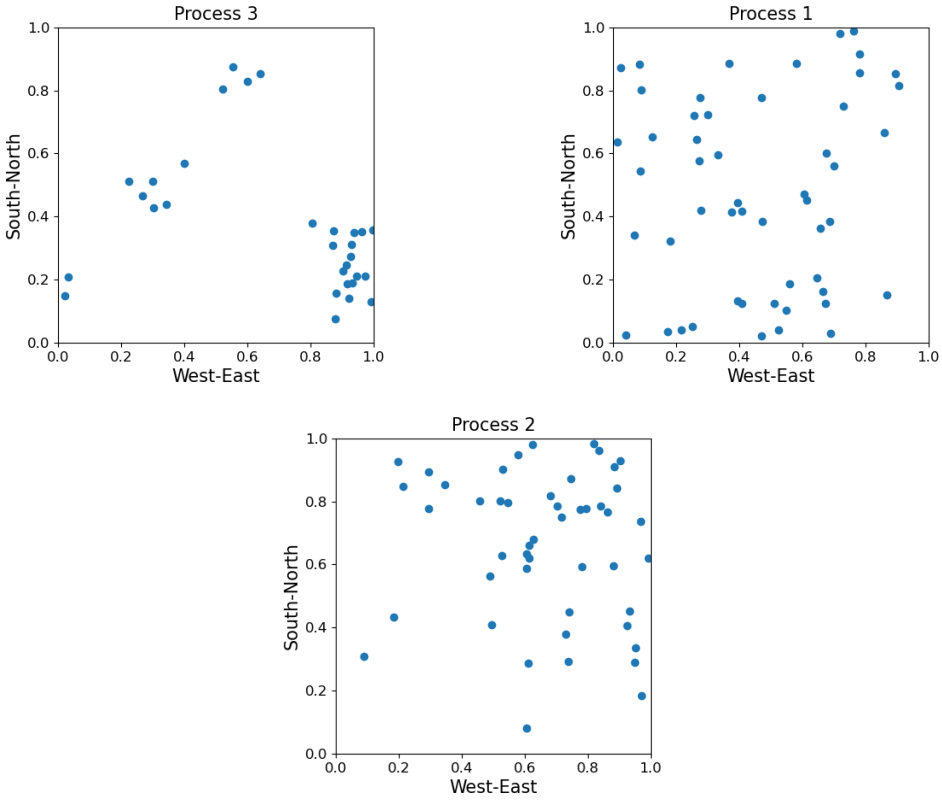


Figure 2: Examples of three point process realizations.

c) Assume that the point process outlined in 2b) describes earthquakes events. But a sensor system only detects an earthquake with probability $\alpha = 0.9$, independently for each event.

In a given year, there are $d = 6$ detected earthquakes in domain D.

Derive the distribution for k given $d = 6$.

Compute the probability that $k \leq 7$ given that $d = 6$.

d) Describe the core elements and properties of a log-Gaussian Cox process model.

Problem 3 Mosaic fields

Consider an Ising type first-order Markov random field $\{l(\mathbf{x}); \mathbf{x} \in \mathcal{D}\}$ with $l(\mathbf{x}) \in \{0, 1\}$, where $\mathcal{D}$ is an $n_1 \times n_2$ lattice with n_1 rows and n_2 columns. Let β be the Ising model interaction parameter on the exponential form.

Figure 3 illustrates a domain with 6 nodes on a 2×3 grid with its clique structure. Let $l_i = l(\mathbf{x}_i)$ be the binary variable at cell i.

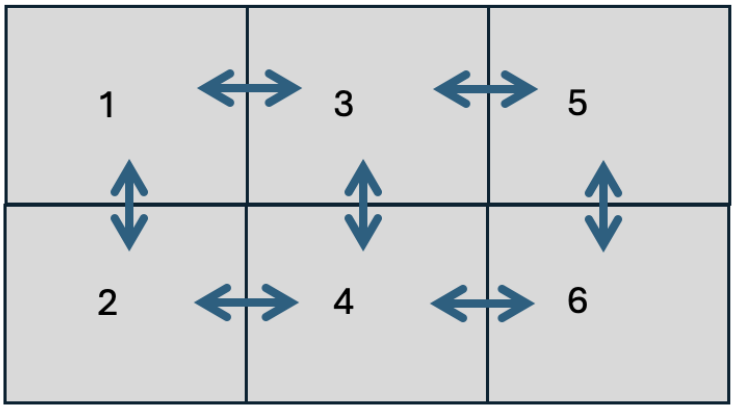


Figure 3: Illustration of a small grid with 6 cells and nearest-neighbor clique interactions (double-edged arrows).

a) Define the joint probability mass function for the binary variables on the grid.

Cell 3 and 4 are known to be white ($l_3 = 1$ and $l_4 = 1$). Setting $\beta = 1$, compute the probability that the entire grid is white.

b) Assume instead that the grid is of size 200×301, and the center column 151 is white.

Describe and provide pseudo-code for using an MCMC sampler to generate realizations of the variables on this grid, given the prior model and the center column configuration.