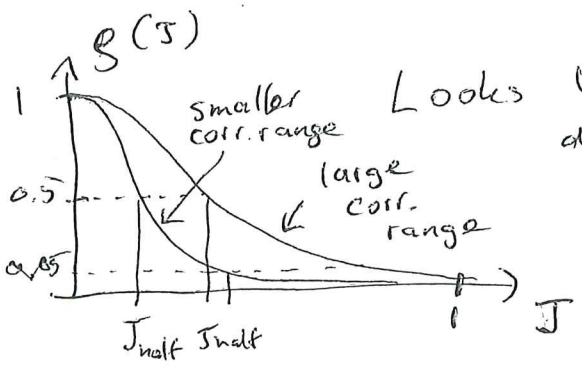
1.

a)



Looks like a Gaussian distribution, for positive distances, and scaled to 1 for $J = 0$.

$\rho(J) = e^{-\xi J^2}$, effective range $\rho(J_{eff}) = 0.05$.

$\rightarrow \quad \xi \cdot J_{eff}^2 = 3, \quad \xi = \frac{3}{J_{eff}^2}$

$\rho(J_{half}) = e^{-\xi J_{half}^2} = 0.5$

$J_{half} = \sqrt{\frac{\ln 0.5}{-\xi}} = 0.14$ for $\xi = \frac{3}{0.3^2} \approx 33$

$= 0.05$ for $\xi = \frac{3}{0.1^2} \approx 300$

$\rho'(J) = -\xi \cdot 2 \cdot J \, e^{-\xi J^2}$

$\rho''(J) = -\xi \cdot 2 \, e^{-\xi J^2} + \xi \cdot \xi \cdot 2 \cdot J \cdot 2 \cdot J \, e^{-\xi J^2}$

$\rho''(J_{curv}) = 0 \quad \rightarrow \quad 1 = 2 \cdot \xi \, J_{curv}^2$

$\frac{1}{\sqrt{2 \cdot \xi}} = J_{curv}$

$J_{curv} = \begin{cases} 0.12 & \text{for } \xi = 33 \\ 0.04 & \text{for } \xi = 300 \end{cases}$

1b)

$r = r(x_0)$, $d_0 = r + \varepsilon_0$, $\varepsilon_0 \sim N(0, 0.1^2)$

$$p(r|d_0) \propto p(r) \cdot p(d_0|r)$$
$$\propto \exp\left(-\frac{1}{2}r^2 - \frac{1}{2}\frac{(d_0 - r)^2}{0.1^2}\right)$$
$$\propto \exp\left(-\frac{1}{2}r^2\underbrace{\left[1 + \frac{1}{0.1^2}\right]}_{\frac{1}{\sigma^2_{r|d_0}}} + r\underbrace{\left[\frac{d_0}{0.1^2}\right]}_{\frac{\mu_{r|d}}{\sigma^2_{r|d_0}}}\right)$$

Quadratic form.

$$\mathrm{Var}(r|d_0) = \sigma^2_{r|d_0} = \frac{1}{1 + \frac{1}{0.1^2}} = \underline{0.099^2}$$

$$\frac{E(r|d_0)}{\mathrm{Var}(r|d_0)} = \frac{d_0}{0.1^2}, \quad E(r|d_0) = \mu_{r|d_0} = \frac{d_0}{0.1} \cdot 0.099^2 = \underline{0.99}$$

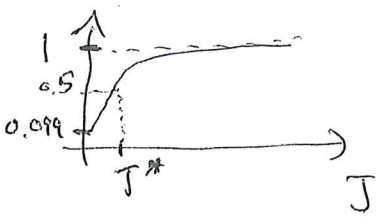
c)
$$\mathrm{Var}(r(x^*)|d_0) = \mathrm{Var}(r(x^*)) - \underbrace{\mathrm{Cov}(r(x), d_0)}_{= \mathrm{Cov}(r(x), r(x_0))} \mathrm{Var}(d_0)^{-1} \mathrm{Cov}(d_0, r(x^*))$$

$\mathrm{Var}(r(x_0)) + \sigma^2_{\varepsilon}$

$$\mathrm{Var}(r(x)|d_0) = 1 - e^{-\xi J^2}\left[1 + 0.1^2\right]^{-1} e^{-\xi J^2}$$
$$= \underline{1 - 0.99\, e^{-2\xi J^2}}$$

Requirement $\mathrm{Var}(r(x)|d_0) = 0.5^2 = 0.25$.

$$1 - 0.99\, e^{-2\xi \cdot J^{*2}} = 0.25$$



$$2\xi J^{*2} = -\ln\left(\frac{0.75}{0.99}\right)$$
$$J^* = \sqrt{\frac{-\ln\left(\frac{0.75}{0.99}\right)}{2\xi}}$$

1d) $E(r(x_0)|\underline{d}) = E(r(x_0)) + Cov(r(x_0), \underline{d}) \cdot Var(\underline{d}) \cdot (\underline{d} - E(\underline{d}))$

$\{ E(r(x_0)) = E(\underline{\varepsilon}(x_i)) = 0 \}$

$Cov(r(x_0), \underline{d}) = Cov(r(x_0), [r(x_1)+\varepsilon(x_1), \ldots \quad r(x_4)+\varepsilon(x_4)])$

$\varepsilon(x_i)$ independent.

$Cov(r(x_0), \underline{d}) = [Cov(r(x_0), r(x_1)), \ldots \quad Cov(r(x_0), r(x_4))]$

$= [e^{-\frac{3}{5}J_1^2}, e^{-\frac{3}{5}J_2^2}, e^{-\frac{3}{5}J_3^2}, e^{-\frac{3}{5}J_4^2}]$

where $J_i = \|x_0 - x_i\|$, $i = 1, 2, 3, 4$.

$Var(\underline{d}) = Var(\underline{r} + \underline{\varepsilon}) = Var(\underline{r}) + Var(\underline{\varepsilon}) + 0$ (↑ indep. εs.)

$$= \begin{bmatrix} 1 & e^{-\frac{3}{5}J_{12}^2} & e^{-\frac{3}{5}J_{14}^2} & \\ & 1 & & \\ & & 1 & e^{-\frac{3}{5}J_{34}^2} \\ & & & 1 \end{bmatrix} + \sigma_{dil}^2 \cdot I_4$$

$J_{ij} = \|x_i - x_j\|$, $i, j = 1, 2, 3, 4$.

Weight are $[w_1, w_2, w_3, w_4] = Cov(r(x_0), \underline{d}) \cdot Var(\underline{d})^{-1}$

(1×4 vector; 4×4 matrix.)

Here:
- Prior model, weights to data less than 1 (sum)
- Point 3 and 4 are close. They should not count twice, but almost like one.
- Alt 2 is right!
- Alt 1 wrong because sum to 1 and not accounting for 3, 4 correlation
- Alt 3 wrong because sum to 1 leaves no trust in prior mean, Gauss random field.

2.

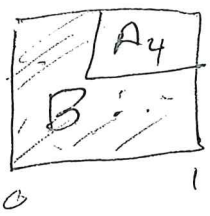
a) Top-left (Proc 3) has clustering. Could be a Neyman-Scott process with 4-5 mothers and about 2-10 children centered at mother with a kernel (std in normal) of about 0.1.
It could also be a realization from a Log-Gaussian Cox process with high variability in latent field.

Topright (Proc 1) appears quite random. Could be a homogeneous Poisson model with $\lambda \simeq 50$ (no. of points).

Bottom (Proc 2) has clear tendency of more point in the north-east. This could be a non-homogeneous Poisson process with an increasing intensity for east and north coordinates. It could also be a clustering process with cluster top right.

Difficult to be sure given only one realization.

2 b)



$$\lambda(x) = 20 x_1 x_2$$

$$\int_0^1 \int_0^1 \lambda(x)\, dx_1, dx_2 = 20 \left[\tfrac{1}{2} x_1^2\right]_0^1 \left[\tfrac{1}{2} x_2^2\right]_0^1$$

$$= 20 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} = \frac{20}{4} = \underline{5}$$

$$E(k) = 5.$$

$$k \sim \text{Poisson}(5).$$

$$P(k \le 2) = \sum_{k=0}^{2} e^{-5} \frac{5^k}{k!} = \underline{0.124}$$

$$k_A \sim \text{Poisson}(\lambda_{A_4}), \quad \lambda_{A_4} = \int_{0.5}^{1}\int_{0.5}^{1} 20 x_1 x_2 \, dx_1 dx_2$$

$$= 20 \left[\tfrac{1}{2} x_1^2\right]_{0.5}^{1} \left[\tfrac{1}{2} x_2^2\right]_{0.5}^{1}$$

$$= \frac{20}{4} \cdot \underbrace{(1 - 0.25)}_{0.56}^2 = \underline{2.81}$$

$$P(k = 5 \mid k_{A4} = 5) = \frac{P(k = 5 \cap k_{A4} = 5)}{P(k_{A4} = 5)}$$

$$= \frac{P(k_B = 0 \cap k_{A4} = 5)}{P(k_{A4} = 5)} \overset{\text{Poisson}}{=} \frac{P(k_B = 0)\, P(k_{A4} =}{P(k_{A4} = 5)}$$

$$= P(k_B = 0)$$

$$k_B \sim \text{Poisson}\left(\underbrace{\tfrac{20}{4} \cdot 0.44}_{2.2}\right) = e^{-2.2} = \underline{0.11}$$

b)

$$P(k_{A_4}=0 \mid k=5) = \binom{5}{0} 0.56^5 \; 0.44^0$$

Binomial

$$= 0.56^5 = \underline{0.055}$$

2c) $P(k \mid d) \propto p(k) \cdot p(d \mid k)$, $k = d, d+1, \ldots$

$$\propto e^{-\lambda_0} \frac{\lambda_0^k}{k!} \binom{k}{d} \alpha^d (1-\alpha)^{k-d}$$

$$\propto \frac{[\lambda_0(1-\alpha)]^{k-d}}{(k-d)!}$$

Poisson distribution, shifted to d from 0.

Intensity $\lambda_0(1-\alpha)$.

$$P(k \le 7 \mid d=6) = \frac{[\lambda_0(1-\alpha)]^0}{0!} e^{-\lambda_0(1-\alpha)} + \frac{[\lambda_0(1-\alpha)]^1}{1!} e^{-\lambda_0(1-\alpha)}$$

$\lambda_0 = 5$
$\alpha = 0.9$
$\lambda_0(1-\alpha) = 5 \cdot (1-0.9) = 0.5$

$$= e^{-\lambda_0(1-\alpha)} \left(1 + \lambda_0(1-\alpha)\right)$$

$$= e^{-0.5}(1+0.5) = 0.91$$

2d) A log-Gaussian Cox process has two building blocks

i) An underlying (latent) Gaussian random field model

$$\{r(x); x \in D\}$$

ii) An nonhomogeneous Poisson process with intensity defined by

$$\lambda(x) = \exp(r(x)).$$

For understanding;

- It is a clustering process, where the independence inherent in Poisson is no longer valid when integrating out the unknown $\{r(x); x \in D\}$.
- The properties are defined via the mean and variance-covariance structure of the $\{r(x); x \in D\}$.

3. a)

$$p(\underline{l}) \propto \exp\Big(\beta \sum_{i\sim j} I(l_i = l_j)\Big)$$

{on exponential form. Alt: $\beta^{*\, I(l_i = l_j)}$ }

$$p(\underline{l}) \propto \exp\Big(\beta\big[I(l_1=l_2)+I(l_1=l_3)+I(l_2=l_4)+I(l_3=l_4) + I(l_3=l_5)+I(l_4=l_6)+I(l_5=l_6)\big]\Big)$$

$$p(\underline{l} \mid l_3=1, l_4=1) \propto \exp\Big(\beta\big[I(l_1=l_2)+I(l_1=1)+I(l_2=1)+ I(l_5=1)+I(l_6=1)+I(l_5=l_6)\big]\Big)$$

l_1, l_2 are independent of l_5, l_6, given l_3, l_4.

$$p(l_1=1, l_2=1 \mid l_3=1, l_4=1) \propto \exp(\beta \cdot 3)$$
$$p(l_1=0, l_2=1 \mid l_3=1, l_4=1) \propto \exp(\beta)$$
$$p(l_1=1, l_2=0 \mid l_3=1, l_4=1) \propto \exp(\beta)$$
$$p(l_1=0, l_2=0 \mid l_3=1, l_4=1) \propto \exp(\beta)$$

$$p(l_1=1, l_2=1 \mid l_3=1, l_4=1) = \frac{\exp(\beta \cdot 3)}{\exp(\beta \cdot 3) + 3 \cdot \exp(\beta)}$$

$$= \underline{0.71}$$

By symmetry/cond independence

$$p(l_1=1, l_2=1, l_5=1, l_6=1 \mid l_3=1, l_4=1) = 0.71 \cdot 0.71 = \underline{0.51}$$

b)

For large grid, the number of configurations gets too large for enumeration and normalization. Instead one must use Monte Carlo methods.

MCMC algorithms include Gibbs-sampling and Metropolis Hastings (M-H). A single-site routine of M-H is provided below.

This is an iterative method for generating samples. Over time it converges to the correct distribution, no matter the starting configuration.

Estimates of statistics (probabilities) can be computed from sample averages over the multiple realizations.

Initiate $\underline{l}^{(0)}$. Set $l_B^{(0)} = 1$ (B is middle column).

Iterate for $b = 1, 2, \ldots B_{max}$

\# sample a random cell in D/B. $\rightarrow i$

\# suggest to change color of cell $\rightarrow$ $l_i^* \rightarrow 0$ if $l_i^{(b-1)} = 1$; $l_i^* \rightarrow 1$ if $l_i^{(b-1)} = 0$

\# compute acceptance rate

$$\alpha = \min(1, a_i)$$

where $a_i = \exp\left(\beta\left[\sum_{i \sim j} I(l_i^* = l_j^{(b-1)}) - \sum_{i \sim j} I(l_i^{(b-1)} = l_j^{(b-1)})\right]\right)$

\# accept or reject

if $\alpha > \text{Unif}(0,1)$

$$l_i^{(b)} = l_i^*$$

else

$$l_i^{(b)} = l_i^{(b-1)}$$

$b = b + 1$