NTNU – Trondheim
Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4250 Spatial Statistics**

Academic contact during examination: Professor Henning Omre

Phone: 90937848

Examination date: June 05. 2015

Examination time (from–to): 09:00-13:00

Permitted examination support material: C:
  Tabeller og Formler i Statistikk, Tapir
  NTNU certified calculator
  Personal, hand written, yellow peep sheet - A5-format

Language: English

Number of pages: 3

Number pages enclosed: 0

Checked by:

Date Signature

Problem 1 CONTINUOUS RANDOM FIELD

Consider a continuous random field $\{R(x); x \in \mathbb{R}^1\}$, defined on a 1D reference space $[-\infty, \infty]$. The model parameters are:

$$\mathrm{E}\{R(x)\} = \mu_R(x) = 10 - (x-a)^2 \; ; x \in \mathbb{R}^1$$
$$\mathrm{Cov}\{R(x'), R(x'')\} = c_R(\tau) = \exp\{-\frac{1}{10}\tau^2\} \; ; x', x'' \in \mathbb{R}^1$$

with $\tau = x'' - x'$.

Hence the field has the highest expected value 10 in the unknown location $x = a$, and the covariance function is given and translation invariant.

a) Specify the requirements for a valid spatial covariance function.

Draw a sketch of the covariance function, $c_R(\tau)$, and explain which characteristics of the field that can be interpreted from the sketch.

Let the random field be observed in the locations $(x_1, \ldots, x_n)$, which entails that the observations are $R_o = (R(x_1), \ldots, R(x_n))$.

b) Define the linear estimator for the location of the highest expected value a:

$$\hat{a} = \sum_{i=1}^{n} \beta_i R(x_i)$$

with $\beta = (\beta_1, \ldots, \beta_n)$ being unknown weights to be determined.

Develop an expression for the best linear unbiased estimator under quadratic loss for a based on the set of observations R_o. Only the minimization problem to be solved in order to identify the weights need to be developed.

Define the associated differential field $\{R^*(x) = \frac{dR(x)}{dx}; x \in \mathbb{R}^1\}$.

c) Develop an expression for the following model parameters of the differential field $\{R^*(x); x \in \mathbb{R}^1\}$:

$$\mathrm{E}\{R^*(x)\} = \mu_{R^*}(x) \; ; x \in \mathbb{R}^1$$
$$\mathrm{Cov}\{R^*(x'), R^*(x'')\} = c_{R^*}(\tau) \; ; x', x'' \in \mathbb{R}^1$$

with $\tau = x'' - x'$.

Draw a sketch of the covariance function, $c_{R^*}(\tau)$, and explain which characteristics of the field that can be interpreted from the sketch.

Assume that the random differential field also is observed in the locations $(x_1, \ldots, x_n)$, hence the differential observations are $R_o^* = (R^*(x_1), \ldots, R^*(x_n))$.

d) Define the linear estimator for the location for the highest expected value a:

$$\hat{a} = \sum_{i=1}^{n} \beta_i R(x_i) + \sum_{i=1}^{n} \gamma_i R^*(x_i)$$

with unknown weights $\beta = (\beta_1, \ldots, \beta_n)$ og $\gamma = (\gamma_1, \ldots, \gamma_n)$ to be determined.

Develop an expression for the best linear unbiased estimator under quadratic loss for a based on the sets of observations R_o and R_o^*. Only the minimization problem to be solved in order to identify the weights need to be developed.

Problem 2 EVENT RANDOM FIELD

Consider a point random field $\mathbb{F}_X : \{X_i; i = 1, \ldots, N; \mathcal{D} \subset \mathbb{R}^2\}$ defined on the area $\mathcal{D} : [0, 10] \times [0, 10]\ [m^2] \in \mathbb{R}^2$.

Assume first that the field is a homogenuous Poisson random field with constant intensity parameter $\lambda \geq 0\ [m^{-2}]$.

a) Specify the probability for zero points in the area $\mathcal{D}$. Justify the answer.

Specify the expected number of points in the area $\mathcal{D}$. Justify the answer.

Given that the number of points in the area $\mathcal{D}$ is $N = 10$, specify the probability for exactly 3 points being in the sub-area $\mathcal{D}_\Delta : [0, 5] \times [0, 5]\ [m^2]$. Justify the answer.

Assume hereafter that the field is a Cox random field where, conditional on the intensity parameter λ, the field is a homogenuous Poisson field as defined above. The intensity parameter λ is now random with probability density $f(\lambda) = \frac{1}{\mu} \exp\{-\frac{\lambda}{\mu}\}$; $\lambda \geq 0$.

b) Develop an expression for expected number of points $\mathrm{E}\{N\}$ and the variance of number of points $\mathrm{Var}\{N\}$ for the Cox field.

Define S as the shortest distance between the centre location in the area $\mathcal{D}$ and the closest point in the point field $\mathbb{F}_X$. Assume that border effects caused by the finite size of the area can be ignored.

c) Develop an expression for the probability density of S for the Cox field.

Problem 3 MOSAIC RANDOM FIELD.

Consider a mosaic random field $L : \{L_x \,; x \in \mathcal{L}_{\mathcal{D}}\}$ where $\mathcal{L}_{\mathcal{D}}$ is a regular grid with n nodes on $\mathcal{D} \subset \mathbb{R}^2$, og $L_x \in \Omega_l : \{W, B\}$. Hence the variable L_x may belong to one of the classes white (W) or black (B) for each $x \in \mathcal{L}_{\mathcal{D}}$.

Define the following Gibbs model for the field:

$$\text{Prob}\{L = l; \beta\} = \text{const}_\beta \times \exp\{\beta \Sigma_{<u,v>} I(l_u = l_v)\}$$

where $u, v \in \mathcal{L}_{\mathcal{D}}$, $< u, v >$ represent all pairs of closest neighbors in the grid $\mathcal{L}_{\mathcal{D}}$ and $I(A)$ is an indicator function taking the value 1 whenever A is true and 0 otherwise. Hence the field is an Ising random field. The model parameter β is assumed known.

Realizations of the field can, theoretically seen, be generated by a single-site Markov chain Monte Carlo (McMC) Metropolis-Hasting (M-H) simulation algorithm. This algorithm is iterative and is based on a proposal- and accept/reject-consept in each iteration. Two possible proposal distributions are: draw a node uniformly over $\mathcal{L}_{\mathcal{D}}$ and update the node-class by either to a) change class or b) generate a class uniformly over Ω_l , independent of the classes in the neighborhood.

a) Use formal algorithm specification and mathematical formalism in the answers, and use precise notation.

Specify a single-site McMC M-H simulation algorithm for the field.

Specify the two proposal distributions outlined above and discuss their characteristics.

Develop computationally efficient expressions for the accept probabilities for the two proposal distributions.

Compare the two algorithms defined from the two proposal distributions by calculating the probabilities for the realizations of the field being changed in an iteration, and discuss the consequences of this for the convergence and mixing characteristics of the algorithms.