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EXAM IN COURSE TMA4250 SPATIAL STATISTICS

Thursday, May 27, 2004
Time: 0900-1300

Hjelpemidler:
- Statistiske tabeller og formler, Tapir
- Godkjent lommekalkulator
- Egetprodusert gult titte-ark - A4-format

Faglærer:
- Prof. Henning Omre, Institutt for matematiske fag; NTNU

Sensuren faller i uke 25.

Problem 1 CONTINUOUS FIELDS

Consider a stationary, isotropic Gaussian random field (RF) $\{R(x); x \in \mathcal{D} \subset \mathbb{R}^2\}$. Assume further that the RF is differentiable. The RF is parametrized by expectation, spatial covariance function and variogram function defined as:

$$\mu = E\{R(x)\}$$

$$c(\Delta x) = \text{Cov}\{R(x'), R(x'')\}$$

$$\gamma(\Delta x) = \frac{1}{2} \text{Var}\{R(x') - R(x'')\}$$

where $\Delta x = |x' - x''|$

a) Develope the relation between $c(\Delta x)$ and $\gamma(\Delta x)$. Explain what is known about $c(\Delta x)$ and $\gamma(\Delta x)$ from the information that the RF is differentiable.

The RF is observed with observation error, once in each of the locations $x_1, \dots, x_n \in \mathcal{D}$:

$$R^0(x_i) = R(x_i) + U_i : i = 1, \dots, n$$

where $U_1, \dots, U_n$ are independent identically Gaussian distributed with expectation 0 and variance σ_U^2, and moreover independent of $\{R(x); x \in \mathcal{D}\}$.

b) Define the following naive estimators for μ and $\gamma(\Delta x)$:

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} R^0(x_i)$$

$$\hat{\gamma}(\Delta x) = \frac{1}{2n_{\Delta x}} \sum_{ij \in A_{\Delta x}} (R^0(x_i) - R^0(x_j))^2$$

with $A_{\Delta x} : \{ij \,|\, |x_i - x_j| = \Delta x; ij = 1, \dots, n\}$ and $n_{\Delta x} = \sharp A_{\Delta x}$. Evaluate the bias properties of these estimators.

Present a simple graphical sketch for $\hat{\gamma}(\Delta x)$ and explain how $\gamma(\Delta x)$ and σ_U^2 can be estimated.

How can $c(\Delta x)$ be estimated?

c) Assume now that $c(\Delta x)$ and σ_U^2 are known, while μ is unknown. Consider an arbitrary location $x_0 \in \mathcal{D}$ and define the predictor:

$$\hat{R}(x_0) = \sum_{i=1}^{n} \alpha_i R^0(x_i)$$

where $\alpha_i; i = 1, \dots, n$ are unknown weights. Develope the ordinary Kriging predictor, ie. the unbiased minimum variance predictor, for $R(x_0)$. Specify the associated prediction variance. Present further a simple graphical sketch for the appearance of the predictor with associated prediction variance in and immediately around an arbitrary observation location x_i.

Problem 2 EVENT FIELDS

Consider a point random field (RF) $\{X_i; i = 1, \dots, N\}$ over the area $\mathcal{D} \subset \mathbb{R}^2$. Assume that a relatively small number of points are identified and that the distance from these points to the closest neighboring point are observed. The observations are $d_1, \dots, d_k$ consisting of distances between closest neighboring points.

Assume that the RF is a homogenuous Poisson RF with intensity λ.

a) The probability density function (*pdf*) for the distance between closest neighboring points, d, under Poisson RF assumptions is on the form:

$$f(d) = c_1 d \exp\{c_2 d^2\}$$

where c_1 and c_2 are two constants possibly dependent on λ.

Develope the exact expression for $f(d)$, ie. determine c_1 and c_2.

b) Develope the maximum likelihood estimator for the intensity λ based on the observations $d_1, \ldots, d_k$. In the development one can consider the observations as independent.

Assume hereafter that the RF is a 'hard-core' Poisson RF with parameters: core-radius a and intensity λ. This entails that there is zero probability for two points to be closer to each other than a and that there is no pairwise interaction between points being farther than a from each other. Note that this RF has several similarities with regular Poisson RF except for the total-repulsive behavior in a disc of radius a around each point.

The *pdf* for the distance between closest neighboring points, d, under 'hard-core' Poisson RF assumptions can be approximated by

$$f(d|d > a)$$

ie. the *pdf* for d under regular Poisson RF assumptions conditional by d being larger than a.

c) Develope the maximum likelihood estimators for the parameters a and λ under this approximative *pdf* based on the observations $d_1, \ldots, d_k$. In the development one can consider the observations as independent.

Problem 3 MOSAIC FIELDS

Consider an Ising random field (RF) $L : \{L_x; x \in \mathcal{L}_\mathcal{D}\}$ where $\mathcal{L}_\mathcal{D}$ is a grid over the area $\mathcal{D} \subset \mathbb{R}^2$, and the sample space is $L_x \in \{-1, 1\}$ for all $x \in \mathcal{L}_\mathcal{D}$.

The simultanous probability distribution can be written as:

$$\text{Prob}\,\{L = l\} = \text{const} \times \exp\{\beta \sum_{x \sim y} I(l_x = l_y)\}$$

with $I(A)$ is identical to 1 if A is true and 0 otherwise, and $x \sim y$ denotes all neighboring grid nodes ie. all horizontal and vertical grid node neighbours. Assume further that the model parameter β is known.

The RF is observed grid node by grid node, but in each node there is probability p for wrong outcome to be observed.

a) Specify a formal expression for the observation likelihood function and the posterior probability distribution.

b) Specify a $MCMC$-algorithm such that a sample from the posterior probability distribution can be generated.