Norwegian University of Science and Technology
Department of Mathematical Sciences

Contact person during exam:
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Exam in TMA4250 SPATIAL STATISTICS

Friday May 21. 2010
Time: 09.00 - 13:00

Support:
Statistiske tabeller og formler, Tapir
NTNU Calculator (HP30S)
Personal handwritten yellow peep-sheet - A4-format

Sensur: Friday June 11. 2010

Problem 1 CONTINUOUS RANDOM FIELDS.

Consider a Gaussian random field $\{R(x); x \in \mathcal{D} \subset \mathbb{R}^2\}$, with expectation function $E\{R(x)\} = \mu_R(x)\ ; x \in \mathcal{D}$, variance function $\text{Var}\{R(x)\} = \sigma_R^2(x)\ ; x \in \mathcal{D}$, and positive definite spatial correlation function $\text{Corr}\{R(x'), R(x'')\} = \rho_R(x', x'')\ ; x', x'' \in \mathcal{D}$. Assume that all model parameters are known.

a) Specify the exact requirements for $\{R(x); x \in \mathcal{D}\}$ to be a Gaussian random field.

Specify the additional requirements for $\{R(x); x \in \mathcal{D}\}$ to be stationary and isotropic.

Consider a related continuous random field:

$$\{S(x) = \sum_{l=1}^{L} B_l g_l(x) + R(x) = g^T(x)B + R(x); x \in \mathcal{D}\}$$

with

$$B = (B_1, \ldots, B_L)^T \quad \rightarrow \quad \text{Gauss}_L(b; 0, \sigma_B^2 I)$$
$$g(x) = (g_1(x), \ldots, g_L(x))^T; x \in \mathcal{D}$$

where model parameter σ_B^2 and functions $g(x)$ are known. Assume further that $\{R(x); x \in \mathcal{D}\}$ is a stationary and isotropic Gaussian random field.

b) Is $\{S(x); x \in \mathcal{D}\}$ a Gaussian random field ? Justify the answer.

Develop expressions for the expectation function $\mu_S(x)$, the variance function $\sigma_S^2(x)$; and the spatial correlation function $\rho_S(x', x'')$ for $\{S(x); x \in \mathcal{D}\}$. The expressions will depend on the model parameters of $\{R(x); x \in \mathcal{D}\}$.

Consider a set of observations of $\{S(x); x \in \mathcal{D}\}$ in locations $x^d = (x_1, \ldots, x_n)$ and denote them $s(x^d) = (s(x_1), \ldots, s(x_n))$.

c) Consider an arbitrary location $x_0 \in \mathcal{D}$.

Develop the expression for:

$$Prob\{S(x_0) > s_0 | S(x^d) = s(x^d)\}$$

Problem 2 EVENT RANDOM FIELDS

A cheese-type contains holes that can be modeled as follows: The hole-centres are located according to a homogeneous Poisson random field (three dimensional) $\{X_i, i = 1, \ldots, N; \mathcal{D} \subset \mathbb{R}^3\}$ with intensity λ ($[dm^3]^{-1}$). The holes are circular with constant radius r (dm). Hence a cheese-unit consists of cheese with density ρ (g/dm^3) and holes (that possibly overlap) that add nothing to the weight.

a) Specify the expression for the probability function for the number of hole-centres inside a cheese-unit of volume 1 dm^3.

b) Consider a cheese-unit of volume 1 dm^3.

Develop an expression for the expected weight of the cheese-unit.

Note that the probability density function for the distance from an arbitrary location in the cheese to the closest hole-centre is:

$$f(d) = 4\lambda\pi d^2 \times \exp\{-\lambda\frac{4}{3}\pi d^3\} \; ; d \geq 0$$

c) Consider an exactly circular hole in the cheese-unit (no overlap with other holes). Develop an expression for the probability density function for the minimum thickness of the cheese between this hole and another hole.

Problem 3 MOSAIC RANDOM FIELDS

Consider an Ising random field $L : \{L_u; u \in \mathcal{L}_\mathcal{D}\}$ where $\mathcal{L}_\mathcal{D}$ is a regular lattice over $\mathcal{D} \subset \mathbb{R}^2$, and $L_u \in \{-1, 1\}$. The clique system consists of all closest neighbors of lattice nodes and the interaction parameter is β. The Gibbs formulation of random field is:

$$Prob\{L = l\} = const \times \exp\{-\beta \times \sum_{\substack{u,v \in \mathcal{L}_\mathcal{D} \\ <u,v>}} l_u l_v\}$$

In the following only lattice nodes in the interior of $\mathcal{L}_\mathcal{D}$ need to be considered - hence border problems can be ignored.

a) Specify the Markov formulation for the random field and specify in particular the associated neighborhood system $\{\delta(u); u \in \mathcal{L}_\mathcal{D}\}$

Observations are available in all lattice nodes $d = \{d_u; u \in \mathcal{L}_\mathcal{D}\}$ and the likelihood model contains blurring:

$$[d_u | L = l] = \frac{1}{5} \sum_{t \in \nu(u)} l_t + e_u$$

with blurring-domain $\nu(u)$ being the lattice node u and the four closest lattice nodes. If $u = (i, j)$ the blurring-domain is $((i, j), (i-1, j), (i, j-1), (i+1, j), (i, j+1))$, ie five lattice nodes. The observation errors e_u are spatially independent.

b) Develop the expression for the Markov formulation for the posterior model, $[L|d]$, and specify in particular the associated neighborhood system.