Norwegian University of Science and Technology
Department of Mathematical Sciences

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Exam in TMA4250 SPATIAL STATISTICS

Friday 24. May 2013
Time: 09.00 - 13:00

Support material:
Statistiske tabeller og formler, Tapir
NTNU calculator (HP30S)
Personal, hand written yellow peep sheet - A5-format

Sensur: Friday 14. June 2013

Problem 1 CONTINUOUS RANDOM FIELDS.

Consider a continuous random field $\{R(x); x \in \mathbb{R}^2\}$, with model parameters: $E\{R(x)\} = \mu_R \,; x \in \mathbb{R}^2$ and $\mathrm{Var}\{R(x)\} = \sigma_R^2 \,; x \in \mathbb{R}^2$. The spatial isotropic correlation function is $\mathrm{Corr}\{R(x'), R(x'')\} = \rho_R(\tau) \,; x', x'' \in \mathbb{R}^2$ with $\tau = |x' - x''|$.

a) Specify the exact requirements for $\rho_R(\tau)$ to be a valid spatial correlation function.

Develop the expression for the associated variogram function $\gamma_R(\tau) = \frac{1}{2}\,\mathrm{Var}\{R(x') - R(x'')\}$.

Define an associated continuous random field $\{S(x) = a + R(x) + U(x)\ ; x \in \mathbb{R}^2\}$ where a is a constant. The continuous random field $\{U(x)\ ; x \in \mathbb{R}^2\}$ is a white-noise field, independent of $R(x)$, with parameters $E\{U(x)\} = 0$, $\text{Var}\{U(x)\} = \sigma_U^2$ og $\text{Corr}\{U(x'), U(x'')\} = I(x' = x'')$ where $I(A)$ is a indicator function with value 1 when A is true and otherwise 0 .

b) Develop expressions for $E\{S(x)\} = \mu_S$, $\text{Var}\{S(x)\} = \sigma_S^2$ and $\text{Corr}\{S(x'), S(x'')\} = \rho_S(\tau)$ expressed by the parameters of the fields $R(x)$ og $U(x)$.

Develop further expressions for the covariance $\text{Cov}\{R(x), S(x)\} = \sigma_{RS}^2$ and the spatial cross-correlation function $\text{Corr}\{R(x'), S(x'')\} = \rho_{RS}(\tau)$.

Consider the following two designs of observations:

$$R^o = (R(x_1^{or}), \ldots, R(x_{nr}^{or})) = (R_1^o, \ldots, R_{nr}^o)$$
$$S^o = (S(x_1^{os}), \ldots, S(x_{ns}^{os})) = (S_1^o, \ldots, S_{ns}^o)$$

The objective is to predict the random field $\{R(x)\ ; x \in \mathbb{R}^2\}$ in an arbitrary location $x_+ \in \mathbb{R}^2$, hence $R_+ = R(x_+)$. Use the linear predictor:

$$\hat{R}_+ = \Sigma_{i=1}^{nr} \alpha_i R_i^o + \Sigma_{j=1}^{ns} \beta_j S_j^o$$

with unknown weights $\alpha = (\alpha_1, \ldots, \alpha_{nr})$ and $\beta = (\beta_1, \ldots, \beta_{ns})$ which must be determined.

Assume that the second-order moment parameters σ_R^2 , σ_U^2 and $\rho_R(\tau)$ are known. The remaining model parameters are unknown.

c) Develop the expressions in the minimization system which defines the weights α and β for the minimum variance, linear, unbiased predictor for R_+. The actual minimization need not be performed.

If the designs of observations for $R(x)$ og $S(x)$ are identical, hence have observations at identical locations, then the minimization system can be simplified. Develop the minimization system for this particular case and comment on the solution.

Problem 2 EVENT RANDOM FIELDS

Consider a homogeneous Poisson random field $\{X_i; i = 1, \ldots, N; \mathcal{D} \subset \mathbb{R}^2\}$ defined over the area $\mathcal{D} : 5 \times 5[m^2]$ in the plane. Let the intensity parameter be $\lambda[m^{-2}]$.

a) Specify the probability distribution for the number of points, N , in $\mathcal{D}$.

If a certain area of $1m^2$ in $\mathcal{D}$ is covered, what is the probability to cover exactly 5 points ?

Given that there are 5 points in the covered area, what is the probability distribution for the number of points in the remaining area - hence the area not being covered ?

An inspector survey the entire area $\mathcal{D}$ in order to register the points, but each point is only registered with probability p. Hence it is probability $(1-p)$ for a point to be overlooked. The registration process is considered to be independent from one point to the other.

b) Given that the number of points in $\mathcal{D}$ is $N = n$, what is the probability distribution for the number of registered points N^o ?

Without knowledge of the total number N, develop an expression for the probability distribution for the number of registered points N^o.

c) Given that the number of registered points is $N^o = n^o$, develop an expression for the probability distribution for the total number of points N.

Are the number of registered points N^o and the number of non-registered points $(N - N^o)$ independent ? Justify the answer.

Comments on the answers given above.

Problem 3 MOSAIC RANDOM FIELDS

Consider a mosaic random field $L : \{L_x\ ; x \in \mathcal{L}_\mathcal{D}\}$ where $\mathcal{L}_\mathcal{D}$ is a regular grid over $\mathcal{D} \subset \mathbb{R}^2$, and $L_x \in \Omega_l : \{W, G, B\}$. Hence the variables L_x can belong to one of the classes white (W), grey (G) or black (B) for each $x \in \mathcal{L}_\mathcal{D}$.

Define the following Gibbs formulation for the field:

$$\text{Prob}\{L = l; \beta = (\beta_W, \beta_G, \beta_B)\} = \text{const}_\beta \times \exp\{\Sigma_{<u,v>} \Sigma_{l_u \in \Omega_l} \beta_{l_u} I(l_u = l_v)\}$$

where $< u, v >$ represents all closest neighbours in the grid $\mathcal{L}_\mathcal{D}$ and $I(A)$ is an indicator function taking the value 1 whenever A is true and 0 otherwise. The model parameters $\beta = (\beta_W, \beta_G, \beta_B)$ are associated with spatial continuity for each of the classes.

a) Develop the corresponding Markov formulation for the random field. Be precise with the notation.

Discuss the most important differences between the Gibbs and the Markov formulations.

Assume that a realization of the field is known $l^o : \{l^o_x \,; x \in \mathcal{L}_\mathcal{D}\}$. This realization shall be used to estimate the model parameters $\beta = (\beta_W, \beta_G, \beta_B)$.

b) Discuss how to perform this estimation in a reliable manner.

Specify the expression for pseudo-likelihood that should be maximized in order to determine β.