NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for **TMA4250 Spatial Statistics**

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NTNU certified calculator
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Problem 1 CONTINUOUS RANDOM FIELDS.

Consider a continuous random field $\{R(x); x \in \mathcal{D} \subset \mathbb{R}^2\}$, with model parameters: $E\{R(x)\} = \mu \,; x \in \mathbb{R}^2$ and $\text{Var}\{R(x)\} = \sigma^2 \,; x \in \mathbb{R}^2$. The spatial isotropic correlation function is $\text{Corr}\{R(x'), R(x'')\} = \rho(\tau) \,; x', x'' \in \mathbb{R}^2$ with $\tau = |x' - x''|$. Let $\mathcal{D} : [0, 10] \times [0, 10] \subset \mathbb{R}^2$.

Let expectation μ be unknown, while variance σ^2 and spatial correlation function $\rho(\tau)$ are known.

Define also the spatial average over $\mathcal{D}$:

$$R_{\mathcal{D}} = \frac{1}{|\mathcal{D}|} \int_{\mathcal{D}} R(u) du$$

with $|\mathcal{D}|$ being the area of $\mathcal{D}$.

a) Develop expressions for $E\{R_{\mathcal{D}}\}$ and $\text{Var}\{R_{\mathcal{D}}\}$.

Consider an arbitrary location $x_o \in \mathcal{D}$, and develop an expression for the covariance $\text{Cov}\{R(x_o), R_{\mathcal{D}}\}$.

Let the random field be observed at locations $x_1, \ldots, x_n \in \mathcal{D}$, hence the observations $[R(x_1), \ldots, R(x_n)]$ are collected.

b) Define the linear estimator for expectation μ:

$$\hat{\mu} = \sum_{i=1}^{n} \beta_i R(x_i)$$

with $\beta = (\beta_1, \ldots, \beta_n)$ being unknown weights to be determined.

Develop an expression for the best linear unbiased estimator under quadratic loss for μ, with associated estimation variance. Only the minimization problem to be solved need to be specified.

c) Define the linear predictor for the spatial average $R_{\mathcal{D}}$:

$$\hat{R}_{\mathcal{D}} = \sum_{i=1}^{n} \alpha_i R(x_i)$$

with $\alpha = (\alpha_1, \ldots, \alpha_n)$ being unknown weights to be determined.

Develop an expression for the best linear unbiased predictor under quadratic loss for $R_{\mathcal{D}}$, with associated prediction variance. Only the minimization problem to be solved need to be specified.

Compare the expressions developed in the current point and point b).

Problem 2 EVENT RANDOM FIELDS

Consider a homogeneous, marked point random field $\{(X_i, L_i); i = 1, \ldots, N; \mathcal{D} \subset \mathbb{R}^2\}$ defined over the area $\mathcal{D} : [0, 10] \times [0, 10]$ $[m^2] \in \mathbb{R}^2$, representing horizontal (parallel to first axis) line-segments of length L centred at location X , see Figure 1.

Let $\{X_i; i = 1, \ldots, N; \mathcal{D} \subset \mathbb{R}^2\}$ be a homogenous Poisson RF with intensity parameter be $\lambda \geq 0$ $[m^{-2}]$, while the line-segment length L has probability density function (pdf) $f(l)$. Moreover, let the random variables X and L be independent.

Note that the line-segments may intersect the vertical (parallel to second axis) boundaries of the domain $\mathcal{D}$.

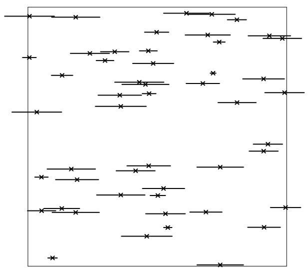

Figure 1: Example of marked point random field.

Assume firstly that the line-segment lengths L are constant equal to 2 $[m]$, ie $f(l)$ is a Dirac pdf at $l = 2$.

a) Specify an expression for the pdf for the number of line-segments in the domain $\mathcal{D}$.

Specify an expression for the pdf for the number of line-segments that intersects the boundary of $\mathcal{D}$.

Specify the expected number of intersections.

b) Consider one specific line-segment located centrally in the domain $\mathcal{D}$, such that boundary effects can be ignored.

Develop an expression for the pdf of the shortest distance between this specific line-segment and any other line-segment.

Assume now that the line-segment lengths are random with pdf:

$$f(l) = \begin{cases} \frac{1}{2}l & \text{for } 0 < l < 2 \\ 0 & \text{otherwise} \end{cases}$$

c) Develop an expression for the pdf for the number of line-segments that intersects the boundary of the domain $\mathcal{D}$.

Consider one arbitrary line-segment which intersects the boundary of $\mathcal{D}$.

Develop an expression for the pdf of the line-segment length of this line-segment that intersects the boundary. Comment on the results.

Problem 3 MOSAIC RANDOM FIELDS

Consider a mosaic random field $L : \{L_x \,; x \in \mathcal{L}_\mathcal{D}\}$ where $\mathcal{L}_\mathcal{D}$ is a regular grid with n nodes over $\mathcal{D} \subset \mathbb{R}^2$, and $L_x \in \Omega_l : \{W, B\}$. Hence the variables L_x can belong to one of the classes white (W) or black (B) for each $x \in \mathcal{L}_\mathcal{D}$.

Define the following Gibbs formulation for the field:

$$\text{Prob}\{L = l; \beta = (\beta_W, \beta_B)\} = \text{const}_\beta \times \exp\{ \quad \beta_W \times \Sigma_u I(l_u = W) + \beta_B \times \Sigma_u I(l_u = B) \\ + \tfrac{1}{2} \times \Sigma_{<u,v>} I(l_u = l_v)\}$$

where $u, v \in \mathcal{L}_\mathcal{D}$ and $< u, v >$ represents all pairs of closest neighbours in the grid $\mathcal{L}_\mathcal{D}$ and $I(A)$ is an indicator function taking the value 1 whenever A is true and 0 otherwise. The $\beta = (\beta_W, \beta_B)$ are unknown model parameters.

a) Present an interpretation of the model parameters $\beta = (\beta_W, \beta_B)$.

Demonstrate that the Gibbs formulation is over-parametrized and actually is only dependent on $\Delta\beta = (\beta_W - \beta_B)$. Be precise with the notation.

b) Develop the corresponding Markov formulation as a function of $\Delta\beta = (\beta_W - \beta_B)$. Be precise with the notation.

Discuss the most important differences between the Gibbs and the Markov formulations.