Norwegian University of
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Department of Mathematical Sciences

Examination paper for **TMA4250 Spatial Statistics**

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Problem 1 CONTINUOUS RANDOM FIELD

Consider a stationary, isotropic Gaussian random field $\{R(x); x \in \mathcal{D} \subset \mathbb{R}^2\}$ parametrized by the expectation μ_r, the variance σ_r^2 and the spatial correlation function $\rho_r(\tau); \tau \in \mathbb{R}^1$. Let the random field be represented in a n-vector $\mathbf{R}$ on a lattice $\mathcal{L}_\mathcal{D}$ over $\mathcal{D}$.

a) Specify the requirements on the model parameters $(\mu_r, \sigma_r^2, \rho_r(\tau))$ for the random field model to be valid.

Let $N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ be a n-dimensional Gaussian pdf with expectation n-vector $\boldsymbol{\mu}$ and covariance $(n \times n)$-matrix $\boldsymbol{\Sigma}$.

b) The n-dimensional pdf for $\mathbf{R}$ is Gaussian $N_n(\boldsymbol{\mu}_r, \boldsymbol{\Sigma}_r)$, also termed prior pdf $f(\mathbf{r})$.

Specify expressions for $(\boldsymbol{\mu}_r, \boldsymbol{\Sigma}_r)$ by using the model parameters $(\mu_r, \sigma_r^2, \rho_r(\tau))$.

Let the random field be observed in a m-vector $\mathbf{D}$ according to the likelihood relation $f(\mathbf{d}|\mathbf{r})$:

$$[\mathbf{D}|\mathbf{R} = \mathbf{r}] = \mathbf{H}\mathbf{r} + \mathbf{U}$$

where $\mathbf{H}$ is an observation $(m \times n)$-matrix and $\mathbf{U}$ is a Gaussian error m-vector $N_m(0\mathbf{i}_m, \sigma_{d|r}^2\mathbf{I}_m)$ independent of $\mathbf{R}$, with $\mathbf{i}_m$ and $\mathbf{I}_m$ being unity m-vector and identity $(m \times m)$-matrix, respectively. Assume that the actual observations are $\mathbf{D} = \mathbf{d}$.

c) Specify the $(n + m)$-dimensional pdf for the joint $[\mathbf{R}, \mathbf{D}]$.

Specify the n-dimensional pdf for the conditional $[\mathbf{R}|\mathbf{D} = \mathbf{d}]$, termed the posterior pdf $f(\mathbf{r}|\mathbf{d})$.

A realization $\mathbf{r}^s$ from the posterior pdf $f(\mathbf{r}|\mathbf{d})$ can be generated by a randomized optimization approach:

- Generate $\mathbf{r}^*$ from $N_n(\boldsymbol{\mu}_r, \boldsymbol{\Sigma}_r)$
- Generate $\mathbf{d}^*$ from $N_m(\mathbf{d}, \sigma_{d|r}^2\mathbf{I}_m)$
- Compute $\mathbf{r}^s = \arg\min_\mathbf{r}\{(\mathbf{d}^* - \mathbf{H}\mathbf{r})[\sigma_{d|r}^2\mathbf{I}_m]^{-1}(\mathbf{d}^* - \mathbf{H}\mathbf{r})^T + (\mathbf{r}^* - \mathbf{r})\Sigma_r^{-1}(\mathbf{r}^* - \mathbf{r})^T\}$

d) Consider the 1-dimensional simplified case with $n = 1$ and $m = 1$, and then the randomized optimization approach reads:

- Generate r^* from $N_1(\mu_r, \sigma_r^2)$
- Generate d^* from $N_1(d, \sigma_{d|r}^2)$
- Compute $r^s = \arg\min_r\{(d^* - hr)[\sigma_{d|r}^2]^{-1}(d^* - hr) + (r^* - r)[\sigma_r^2]^{-1}(r^* - r)\}$

Demonstrate that the approach does generate a realization from the posterior pdf $f(r|d)$.

Problem 2 EVENT RANDOM FIELD

Consider a homogenuous Poisson point random field $\{X_i; i = 1, \ldots, N; \mathcal{D} \subset \mathbb{R}^2\}$ with model intensity parameter $\lambda \geq 0$. Define an arbitrary sub-domain $\mathcal{B} \subset \mathcal{D}$ and let $\mathcal{N}(\mathcal{B}) \in \{0, 1, \ldots\}$ be the number of points in the sub-domain $\mathcal{B}$.

a) Specify the expression for $\text{Prob}(\mathcal{N}(\mathcal{B}) > 0)$.

Develop expressions for $\text{E}(\mathcal{N}(\mathcal{B})|\mathcal{N}(\mathcal{B}) > 0)$ and $\text{Var}(\mathcal{N}(\mathcal{B})|\mathcal{N}(\mathcal{B}) > 0)$.

Consider two arbitrary sub-domains $\mathcal{B}_1 \subset \mathcal{D}$ and $\mathcal{B}_2 \subset \mathcal{D}$.

b) Specify the expressions for $\text{E}(\mathcal{N}(\mathcal{B}_1))$, $\text{E}(\mathcal{N}(\mathcal{B}_2))$, $\text{Var}(\mathcal{N}(\mathcal{B}_1))$ and $\text{Var}(\mathcal{N}(\mathcal{B}_2))$. Develop the expression for $\text{Cov}(\mathcal{N}(\mathcal{B}_1), \mathcal{N}(\mathcal{B}_2))$

Insert in the answer $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$ and $\mathcal{B}_1 = \mathcal{B}_2$ and discuss the results.

c) Develop the expression for $\text{Prob}(\mathcal{N}(\mathcal{B}_2) = i|\mathcal{N}(\mathcal{B}_1) = j)$.

Insert in the answer $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$ and discuss the results.

Problem 3 MOSAIC RANDOM FIELD.

Consider a mosaic random field $L : \{L_x\ ; x \in \mathcal{L}_\mathcal{D}\}$ where $\mathcal{L}_\mathcal{D}$ is a regular grid with n nodes on $\mathcal{D} \subset \mathbb{R}^2$, and $L_x \in \Omega_l : \{W, B\}$. Hence the variable L_x may belong to one of the classes white (W) or black (B) for each $x \in \mathcal{L}_\mathcal{D}$.

Define the following Gibbs model for the field:

$$\text{Prob}\{L = l; \beta\} = \text{const}_\beta \times \exp\{\beta \Sigma_{<u,v>} I(l_u = l_v)\}$$

where $u, v \in \mathcal{L}_{\mathcal{D}}$, $< u, v >$ represent all pairs of closest neighbors in the grid $\mathcal{L}_{\mathcal{D}}$ and $I(A)$ is an indicator function taking the value 1 whenever A is true and 0 otherwise. Hence the field is an Ising random field. The model parameter $\beta > 0$ is assumed known.

Consider the observations, $O : \{O_x; x \in \mathcal{L}_{\mathcal{D}}\}$ where $O_x \in \{W, B\}$, hence we observe a binary field. Let the likelihood model be:

$$\text{Prob}\{O = o|L = l; p\} = \prod_{x \in \mathcal{L}_{\mathcal{D}}} \text{Prob}\{O_x = o_x|L_x = l_x; p\}$$

where o_x coincide with l_x with probability $(1 - p)$ and is misclassified to the other class with probability p. The model parameter $0 < p < 1$ is assumed known.

a) Demonstrate that the Markov form of the posterior model is:

$$\begin{aligned} &\text{Prob}\{L_x = l_x|O = o, L_{-x} = l_{-x}; \beta, p\} \\ &\quad = \text{const}_{\beta,p} \times \exp\{\ln[\frac{1-p}{p}]I(o_x = l_x) + \beta\Sigma_{<x,v>}I(l_x = l_v)\}; \forall x \in \mathcal{L}_{\mathcal{D}} \end{aligned}$$