NTNU
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Department of Mathematical Sciences

Examination paper for **TMA4250 Spatial Statistics**

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Problem 1 CONTINUOUS RANDOM FIELD

Consider a one-dimensional Gaussian random field $\{r(x); x \in \mathtt{D} \subset \mathbb{R}\}$ parametrized by

$$\begin{aligned} \mathrm{E}\{r(x)\} &= \mu_0 + \mu_1 g(x) \\ \mathrm{Var}\{r(x)\} &= \sigma^2 h(x) \\ \mathrm{Corr}\{r(x'), r(x'')\} &= \rho(x' - x'') \end{aligned}$$

where $\{g(x); x \in \mathtt{D}\}; g(x) \in \mathbb{R}$ and $\{h(x); x \in \mathtt{D}\}; h(x) \in \mathbb{R}_\oplus$ are known functions on $\mathtt{D}$. The other model parameters are $\mu_0, \mu_1 \in \mathbb{R}, \sigma^2 \in \mathbb{R}_\oplus$ and $\rho(\tau) \in [-1, 1]; \tau \in \mathbb{R}$. Assume that the model parameters σ^2 and $\rho(\tau)$ are known, while μ_0, μ_1 are unknown.

a) Specify the mathematical requirements for $\rho(\tau)$ to be a valid spatial correlation function.

Assume that the Gaussian random field over $\mathtt{D}$ is exactly observed by $[r(x_1), r(x_2), \ldots, r(x_n)]$.

b) Consider an arbitrary location $x_0 \in \mathtt{D}$, and define the linear predictor,

$$\hat{r}(x_0) = \sum_{i=1}^{n} \alpha_i r(x_i)$$

with unknown weights $[\alpha_1, \alpha_2, \ldots, \alpha_n]$ to be determined.

Develop the expression for the minimization system to be solved in order to determine the weights for the best linear unbiased predictor (BLUP) under squared error loss.

c) Consider the linear estimators for the unknown model parameters,

$$\begin{aligned} \hat{\mu}_0 &= \sum_{i=1}^{n} \beta_i^0 r(x_i) \\ \hat{\mu}_1 &= \sum_{i=1}^{n} \beta_i^1 r(x_i) \end{aligned}$$

with unknown weights $[\beta_1^0, \beta_2^0, \ldots, \beta_n^0]$ and $[\beta_1^1, \beta_2^1, \ldots, \beta_n^1]$ to be determined.

Develop the expressions for the two minimization systems to be solved in order to obtain the best linear unbiased estimators (BLUE) under squared error loss for μ_0 and μ_1, respectively.

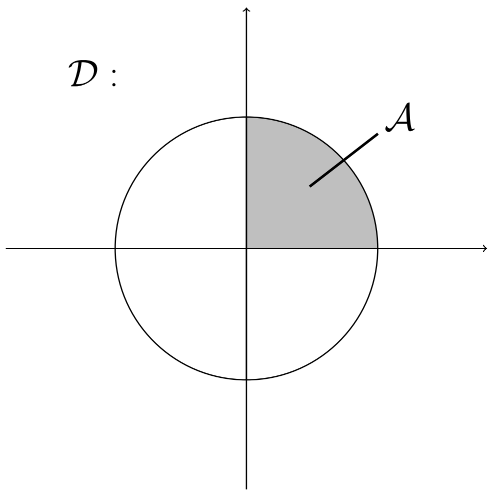

Figure 1: Domains.

Problem 2 EVENT RANDOM FIELD

Consider a stationary Poisson point random field represented by $\{x_i; i = 1, \ldots, n\}; x_i \in \mathtt{D} \subset \mathbb{R}^2; n \in \mathbb{N}_\oplus$ with model intensity parameter $\lambda \geq 0$. Let the domain $\mathtt{D}$ be a circular disc centred in origin location $(0, 0)$ with radius r.

Define a sub-domain $\mathtt{A} \subset \mathtt{D}$, where $\mathtt{A}$ is the upper-right quarter-sector of the disc $\mathtt{D}$, see Figure 1.

Let $k_\mathtt{D} \in \mathbb{N}_\oplus$ and $k_\mathtt{A} \in \mathbb{N}_\oplus$ denote the number of points in the domains $\mathtt{D}$ and $\mathtt{A}$, respectively.

a) Develop expressions for $\mathrm{E}\{k_\mathtt{D}\}, \mathrm{E}\{k_\mathtt{A}\}, \mathrm{Var}\{k_\mathtt{D}\}, \mathrm{Var}\{k_\mathtt{A}\}$ and $\mathrm{Cov}\{k_\mathtt{D}, k_\mathtt{A}\}$.

b) Assume firstly that $k_\mathtt{D}$ is unknown, and that one has observed $k_\mathtt{A} = k$. Specify the expression for $\mathrm{Prob}\{k_\mathtt{D} = i | k_\mathtt{A} = k\}, i \in \mathbb{N}_\oplus$.

Assume secondly that $k_\mathtt{A}$ is unknown, and that one has observed $k_\mathtt{D} = k$. Specify the expression for $\mathrm{Prob}\{k_\mathtt{A} = i | k_\mathtt{D} = k\}, i \in \mathbb{N}_\oplus$.

c) Assume here that $k_\mathtt{D}$ is unknown, and that one has observed $k_\mathtt{A} = k \geq 1$.

Consider the center location in $\mathtt{D}$, being the origin, and define d to be the distance from this center location to the closest point in the point random field.

Develop the expression for the pdf $p(d | k_\mathtt{A} = k); d \in \mathbb{R}_\oplus$.

Problem 3 MOSAIC RANDOM FIELD.

Consider a one-dimensional Markov random field (Markov random profile) $\{l_x\ ; x \in \mathcal{L}_{\mathtt{D}}\}$ where $\mathcal{L}_{\mathtt{D}}$ is a regular grid with n nodes on $\mathtt{D} \subset \mathbb{R}$, represented by the n-vector $\mathbf{l} = (l_1, l_2, \ldots, l_n)$. Let $l_x \in \Omega_l : \{W, B\}$. Hence the variable l_x belongs to one of the classes white (W) or black (B) for each $x \in \mathcal{L}_{\mathtt{D}}$.

Define the following Gibbs formulation for the profile:

$$\begin{aligned} p(\mathbf{l}) &= const \times \prod_{<u,v>} \beta^{I(l_u = l_v)} \\ &= const \times \prod_{i=1}^{n-1} \beta^{I(l_i = l_{i+1})} \end{aligned}$$

where $u, v \in \mathcal{L}_{\mathtt{D}}$, $< u, v >$ represent all pairs of closest neighbors in the grid $\mathcal{L}_{\mathtt{D}}$ and $I(A)$ is an indicator function taking the value 1 whenever A is true and 0 otherwise. Hence the profile is an Ising random profile. The model parameter $\beta \geq 1$ is assumed known.

a) Develop the expression for the Markov formulation of the random profile, ie $p(l_i|\mathbf{l}_{-i}); i = 1, 2, \ldots, n$. Be aware of the boundary expressions.

b) Any pdf for a multivariate random variable can be sequentially decomposed,

$$p(\mathbf{l}) = p(l_1) \prod_{i=2}^{n} p(l_i|l_{i-1}, \ldots, l_1)$$

Develop the expressions for $p(l_i|l_{i-1}, \ldots, l_1); i = 2, 3, \ldots, n$ based on the Gibbs formulation of the random profile. Demonstrate that this sequential decomposition defines a first-order Markov chain along the profile.