Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4250 Spatial Statistics**

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Date Signature

Problem 1 CONTINUOUS RANDOM FIELD

Consider a one-dimensional Gaussian random field $\{r(x); x \in \mathtt{D} \subset \mathbb{R}\}$ parametrized by

$$\begin{aligned} \mathrm{E}\{r(x)\} &= \mu_r \\ \mathrm{Var}\{r(x)\} &= \sigma_r^2 \\ \mathrm{Corr}\{r(x'), r(x'')\} &= \rho_r(\tau) \end{aligned}$$

The model parameters are $\mu_r \in \mathbb{R}, \sigma_r^2 \in \mathbb{R}_{\oplus}$ and $\rho_r(\tau) \in [-1, 1]; \tau = x' - x'' \in \mathbb{R}$.

Assume that the model parameters σ_r^2 and $\rho_r(\tau)$ are known, while μ_r is unknown.

a) Specify the mathematical requirements for $\rho_r(\tau)$ to be a valid positive definite spatial correlation function.

Assume that $\rho_r^i(\tau); i = 1, \ldots, n_\rho$ are positive definite correlation functions. Specify two ways of combining these functions which ensures positive definiteness of their combinations.

Define a supplementary one-dimensional Gaussian random field $\{s(x); x \in \mathtt{D} \subset \mathbb{R}\}$ where

$$[s(x)|r(x)] = \gamma_{sr} r(x) + \epsilon(x); x \in \mathtt{D}$$

with the Gaussian random field $\{\epsilon(x); x \in \mathtt{D} \subset \mathbb{R}\}$ parametrized by

$$\begin{aligned} \mathrm{E}\{\epsilon(x)\} &= 0 \\ \mathrm{Var}\{\epsilon(x)\} &= \sigma_\epsilon^2 \\ \mathrm{Corr}\{\epsilon(x'), \epsilon(x'')\} &= \rho_\epsilon(\tau) \\ \mathrm{Corr}\{\epsilon(x'), r(x'')\} &= 0 \end{aligned}$$

The model parameters are $\gamma_{sr} \in \mathbb{R}$, $\sigma_\epsilon^2 \in \mathbb{R}_{\oplus}$ and $\rho_\epsilon(\tau) \in [-1, 1]; \tau = x' - x'' \in \mathbb{R}$.

Assume that the model parameters γ_{sr}, σ_ϵ^2 and $\rho_\epsilon(\tau)$ are known.

b) Develop expressions for the parameters of the joint Gaussian random field $\{[r(x), s(x)]; x \in \mathtt{D} \subset \mathbb{R}\}$.

Consider the exact observations of the random fields,

$$\mathbf{r}^o = \{r(x_1^{or}), r(x_2^{or})\}$$
$$\mathbf{s}^o = \{s(x_1^{os}), s(x_2^{os})\}$$

where the corresponding four locations in the domain $\mathtt{D}$ may be different.

c) Consider an arbitrary location $x_0 \in \mathtt{D}$, and define the linear predictor,

$$\hat{r}(x_0) = \sum_{i=1}^{2} \alpha_i r(x_i^{or}) + \sum_{i=1}^{2} \beta_i s(x_i^{os})$$

with unknown weights $\{[\alpha_1, \alpha_2], [\beta_1, \beta_2]\}$ to be determined.

Develop the expression for the minimization system to be solved in order to determine the weights for the best linear unbiased predictor (BLUP) under squared error loss. Note that the actual minimization need not be made.

Problem 2 EVENT RANDOM FIELD

One evening, a student-couple buys a circular pizza $\mathtt{P}$ with radius r, hence with area $a_P = \pi r^2$. On top of the pizza pieces of olives are distributed. Denote the locations the olive-pieces on $\mathtt{P}$ by $\mathbf{X}_{\mathtt{P}} : \{x_i; i = 1, \ldots, k_P\}; x_i \in \mathtt{P} \subset \mathbb{R}^2; k_P \in \mathbb{N}_{\oplus}$ with k_P and x_i being the number and center locations of the olive-pieces respectively.

Assume that $\mathbf{X}_{\mathtt{P}}$ is distributed according to a stationary Poisson random field with intensity parameter λ_o, hence both k_P and x_i are random variables.

a) Specify the expression for the pdf of the olive-piece locations on top of the pizza $\mathtt{P}$, $p(x_1, \ldots, x_{k_P})$.

b) Specify an expression for the expected number of olive-pieces on the pizza.

Given that one half of the pizza contains exactly k_h olive-pieces - specify an expression for the expected number of olive-pieces on the pizza.

The student-couple always share the pizza by splitting it into two slices, $\mathtt{P}_M$ and $\mathtt{P}_F$, one for the male and one for the female student. Let the respective pizza areas be a_M and a_F, hence $a_P = a_M + a_F$, and the respective proportions of the pizza are $\nu_M = \frac{a_M}{a_P}$ and $\nu_F = \frac{a_F}{a_P}$.

Experience from the student-couple relationship tells that these proportions are distributed according to the pdf,

$$p(\nu_M, \nu_F; \alpha, \beta) = const \times \nu_M^{\alpha-1} \nu_F^{\beta-1}$$

with $\nu_m + \nu_F = 1, \nu_M \in \mathbb{R}_{[0,1]}, \nu_F \in \mathbb{R}_{[0,1]}$, and model parameters $\alpha \in \mathbb{R}_+$ and $\beta \in \mathbb{R}_+$.

The expected areas of $\mathtt{P}_M$ and $\mathtt{P}_F$ will then be $\frac{\alpha}{\alpha+\beta} a_P$ and $\frac{\beta}{\alpha+\beta} a_P$, respectively. Since the male student usually eats more than the female one, one has $\alpha > \beta$.

c) This evening, after having split the pizza into two slices, $\mathtt{P}_M$ and $\mathtt{P}_F$, the students count the number of olive-pieces on each pizza-slice and observe exactly k_M and k_F.

Given the olive-piece counts, k_M and k_F, and the experience from previous pizza-evenings - develop expressions for the expected areas of the pizza-slices $\mathtt{P}_M$ and $\mathtt{P}_F$ this very evening.

Problem 3 MOSAIC RANDOM FIELD.

Consider a two-dimensional Markov random field $\{l_x \,; x \in \mathcal{L}_{\mathtt{D}}\}$ where $\mathcal{L}_{\mathtt{D}}$ is a regular lattice with n nodes over the domain $\mathtt{D} \subset \mathbb{R}^2$, represented by the n-vector $\mathbf{l}$. Let $l_x \in \Omega_l : \{W, B\}$. Hence the variable l_x belongs to one of the classes white (W) or black (B) for each $x \in \mathcal{L}_{\mathtt{D}}$.

Define the following Gibbs formulation for the random field:

$$p(\mathbf{l}) = const \times \prod_{<u,v>} \beta^{I(l_u = l_v)}$$

where $u, v \in \mathcal{L}_{\mathtt{D}}$, $< u, v >$ represent all pairs of closest neighbors in the grid $\mathcal{L}_{\mathtt{D}}$ and $I(A)$ is an indicator function taking the value 1 whenever A is true and 0 otherwise. Hence the field is an Ising random field. The model parameter $\beta \in \mathbb{R}_{[1,\infty)}$ is assumed known.

Moreover, let the observations be $\{d_x \,; x \in \mathcal{L}_{\mathtt{D}}\}; d_x \in \mathbb{R}$, represented by the n-vector $\mathbf{d}$.

a) Consider a likelihood function being conditional independent with single-site response,

$$p(\mathbf{d}|\mathbf{l}) = \prod_{x \in \mathcal{L}_\mathtt{D}} p(d_x|\mathbf{l}) = \prod_{x \in \mathcal{L}_\mathtt{D}} p(d_x|l_x)$$

Develop the expression for the Gibbs and Markov formulations of the posterior random field.

b) Consider a likelihood function being conditional independent with 5-site response, socalled blurring. The response sites of d_x are location x and the four closest sites, ie a cross of five sites, denoted o_x. The likelihood function is then,

$$p(\mathbf{d}|\mathbf{l}) = \prod_{x \in \mathcal{L}_\mathtt{D}} p(d_x|\mathbf{l}) = \prod_{x \in \mathcal{L}_\mathtt{D}} p(d_x|l_y; y \in o_x)$$

Develop the corresponding expression for the Gibbs and Markov formulations of the posterior random field.

Specify the neighborhood of the Markov form graphically.