SUGGESTED SOLUTION
Exam TMA4250 Spatial Statistics
June 5. 2015
Resp. H. OMRE

Problem 1 Continuous RF

$$R(x) = 10 - (x-a)^2 + R_s(x)$$

a: unknown

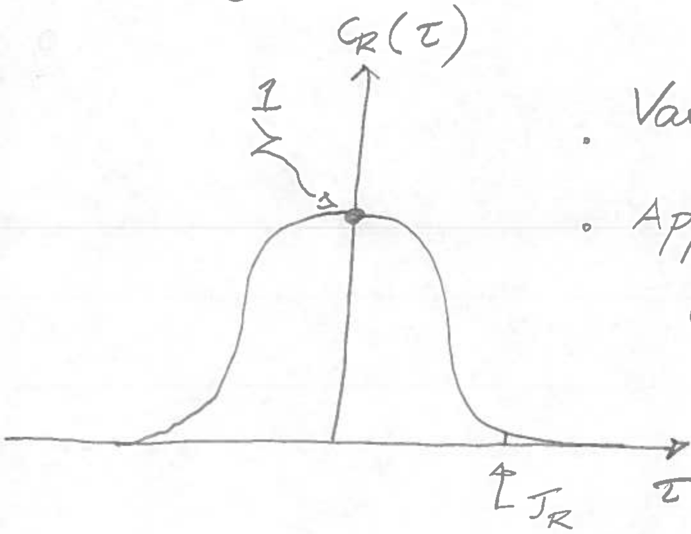
$E\{R_s\} = 0$

$\operatorname{Cov}\{R(x'), R(x'')\} = c_R(\tau) = \exp\{-\frac{1}{10}\tau^2\}$

$\tau = x'' - x'$

a) Spatial covariance function must be positiv definite:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j \, c_R(x_i, x_j) \geq 0 \quad \text{all } \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n, \text{ all } n \geq 2$$



- $\operatorname{Var}\{R_s(x)\} = c_R(0) = 1.0$
- Apparent range: τ_R

$$c_R(\tau_R) = \exp\{-\tfrac{1}{10}\tau_R^2\} = 0.05$$

$$\Downarrow$$

$$\tau_R = [-10 \ln 0.05]^{1/2}$$

Distance to obtain approx non-correlation in the RF

- $\frac{d^n}{d\tau^n} c_R(\tau)\Big|_{\tau=0}$ - number of times differentiality of $c_R(\tau)$ at $\tau = 0$.

$c_R(\tau)$ is infinitely many times differential
$\Rightarrow R(x)$ is "very smooth".

Observations $R_0 : \{R(x_1), \dots, R(x_n)\}$

b) Define linear estimator for a:

$$\hat{a} = \sum_{i=1}^{n} \beta_i R(x_i) \; ; \; \beta = (\beta_1, \dots, \beta_n) \text{ - to be determined}$$

Unbiasedness:

$$E\{\hat{a}\} = \sum_i \beta_i E\{R(x_i)\} = \sum_i \beta_i [10 - (x_i - a)^2]$$

$$= \sum_i \beta_i [10 - (x_i^2 - 2a x_i + a^2)]$$

$$= 10 \sum_i \beta_i - \sum_i \beta_i x_i^2 + 2a \sum \beta_i x_i - a^2 \sum_i \beta_i$$

with

$$\left.\begin{aligned} &\sum \beta_i = 0 \\ &\sum \beta_i x_i = \frac{1}{2} \\ &\sum \beta_i x_i^2 = 0 \end{aligned}\right\} \Rightarrow E\{\hat{a}\} = a$$

Variance:

$$Var\{\hat{a} - a\} = Var\{\hat{a}\} = \sum_i \sum_j \beta_i \beta_j Cov\{R(x_i), R(x_j)\}$$

$$= \sum_i \beta_i^2 + \sum_i \sum_{i \neq j} \beta_i \beta_j \exp\left\{-\frac{1}{10}(x_i - x_j)^2\right\}$$

Minimation to be solved - by lagrange opt:

$$\hat{\beta} = \operatorname{argmax}\{Var(\hat{a} - a)\}$$

$$\sum \beta_i = 0$$

$$\sum \beta_i x_i = \frac{1}{2}$$

$$\sum \beta_i x_i^2 = 0$$

Define: $\left\{ R^*(x) = \frac{d}{dx} R(x) \,;\, x \in \mathbb{R}^1 \right\}$

c)

$$R^*(x) = \frac{d}{dx} R(x) = \frac{d}{dx}\left[10 - (x-a)^2 + R_s(x) \right]$$

$\frac{d}{dx}$ - lin. oper.

$$= -2(x-a) + \frac{d}{dx} R_s(x)$$

$$E\{R^*(x)\} = -2(x-a) + \frac{d}{dx} \underbrace{E\{R_s(x)\}}_{=0} = -2(x-a)$$

$$\mathrm{Cov}\{R^*(x'), R^*(x'')\} = \frac{d}{dx'}\left[\frac{d}{dx''} \mathrm{Cov}\{R(x'), R(x'')\} \right]$$

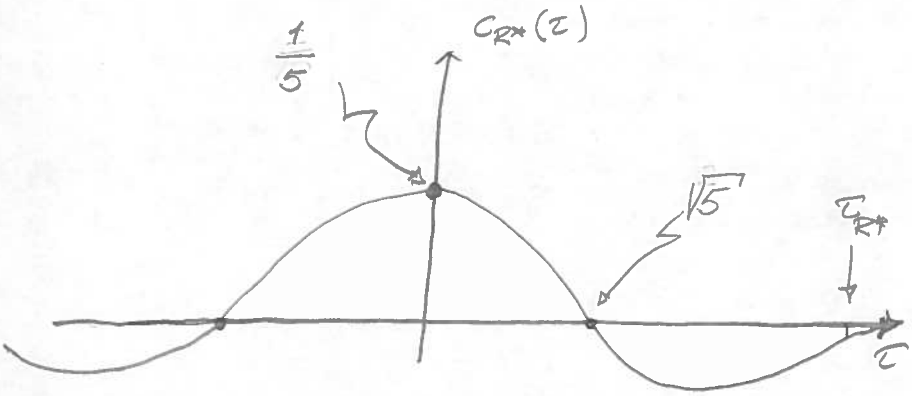
$$= \frac{d}{dx'}\left[\frac{d}{dx''} \exp\left\{ -\frac{1}{10}(x''-x')^2 \right\} \right]$$

$$= \frac{d}{dx'}\left[-\frac{2}{10}(x''-x') \exp\left\{ -\frac{1}{10}(x''-x')^2 \right\} \right]$$

$$= \left[\frac{2}{10} - \left(\frac{2}{10}\right)^2 (x''-x')^2 \right] \exp\left\{ -\frac{1}{10}(x''-x')^2 \right\}$$

$$= \frac{1}{5}\left[1 - \frac{1}{5}(x''-x')^2 \right] \exp\left\{ -\frac{1}{10}(x''-x')^2 \right\}$$

NOTE: $\mathrm{Var}\{R^*(x)\} = \mathrm{Cov}\{R^*(x), R^*(x)\} = \frac{2}{10} = \frac{1}{5}$



- Apperent range J_{R^*}

$$C_{R^*}(\tau_{R^*}) = \frac{1}{5}\left[1 - \frac{\tau_{R^*}^2}{5} \right] \exp\left\{ -\frac{1}{10}\tau_{R^*}^2 \right\} = -0.05$$

$\Downarrow$ num

$\tau_{R^*} =$

- $R^*(x)$ is 'very smooth'.

Observations $R_0^* = (R^*(x_1), \dots, R^*(x_n))$
in same location as for R_0.

d) Define linear estimator for a:

$$\hat{a} = \sum_{i=1}^{n} \beta_i R(x_i) + \sum_{i=1}^{n} \gamma_i R^*(x_i) \qquad \begin{matrix} \beta = (\beta_1, \dots, \beta_n) \\ \gamma = (\gamma_1, \dots, \gamma_n) \end{matrix} \text{ to be determined}$$

Unbiasedness:

$$E\{\hat{a}\} = \sum_i \beta_i E\{R(x_i)\} + \sum_i \gamma_i E\{R^*(x_i)\}$$

$$= \sum_i \beta_i [10 - x_i^2 + 2ax_i - a^2] + \sum_i \gamma_i [-2x_i + 2a]$$

$$\begin{aligned} &= [10 \textstyle\sum \beta_i - \sum \beta_i x_i^2 - 2\sum \gamma_i x_i] && \to 0 \\ &+ [2\textstyle\sum \gamma_i + 2\sum \beta_i x_i]\, a && \to a \\ &+ [-\textstyle\sum \beta_i]\, a^2 && \to 0 \end{aligned} \Rightarrow \begin{cases} \sum \beta_i x_i^2 + 2\sum \gamma_i x_i = 0 \\ \sum \gamma_i + \sum \beta_i x_i = \frac{1}{2} \\ \sum \beta_i = 0 \end{cases}$$

Variance:

$$\begin{aligned} Var\{\hat{a} - a\} = Var\{\hat{a}\} = &\sum_i \sum_j \beta_i \beta_j Cov\{R(x_i), R(x_j)\} \\ &+ \sum_i \sum_j \gamma_i \gamma_j Cov\{R^*(x_i), R^*(x_j)\} \\ &+ \sum_i \sum_j \beta_i \gamma_j Cov\{R(x_i), R^*(x_j)\} \\ &+ \sum_i \sum_j \gamma_i \beta_j Cov\{R^*(x_i), R(x_j)\} \end{aligned}$$

Need to determine cross-covariance functions:

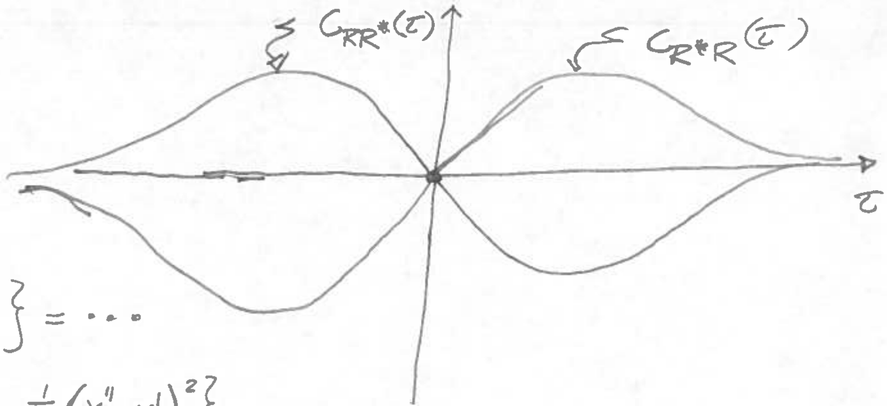
$Cov\{R(x_i), R^*(x_j)\}$

$Cov\{R^*(x_i), R(x_j)\}$

$$C_{RR^*}(\tau) = \mathrm{Cov}\left\{R(x'), \overset{*}{R}(x'')\right\} = \mathrm{Cov}\left\{R(x'), \frac{d}{dx''}R(x'')\right\}$$

$$= \frac{d}{dx''}\mathrm{Cov}\left\{R(x'), R(x'')\right\}$$

$$= \frac{d}{dx''}\exp\left\{-\frac{1}{10}(x''-x')^2\right\} = -\frac{2}{10}(x''-x')\exp\left\{-\frac{1}{10}(x''-x')^2\right\}$$



$$C_{R^*R}(\tau) = \mathrm{Cov}\left\{\overset{*}{R}(x'), R(x'')\right\} = \ldots$$

$$= \frac{2}{10}(x''-x')\exp\left\{-\frac{1}{10}(x''-x')^2\right\}$$

Hence:

$$C_{RR^*}(\tau) = -C_{R^*R}(\tau)$$

$$C_{RR^*}(0) = C_{R^*R}(0) = 0 = \mathrm{Cov}\left\{R(x), \overset{*}{R}(x)\right\}$$

- obs & deriv in same loc are uncorrelated!

Variance

$$\mathrm{Var}\{\hat{a}-a\} = \sum_i\sum_j \beta_i\beta_j \exp\left\{-\frac{1}{10}(x_j-x_i)^2\right\}$$
$$+ \sum_i\sum_j \gamma_i\gamma_j \frac{1}{5}\left[1-\frac{1}{5}(x_j-x_i)^2\right]\exp\left\{-\frac{1}{10}(x_j-x_i)^2\right\}$$
$$+ \sum_i\sum_j \beta_i\gamma_j\left[-\frac{1}{5}(x_j-x_i)\exp\left\{-\frac{1}{10}(x_j-x_i)^2\right\}\right]$$
$$+ \sum_i\sum_j \gamma_i\beta_j \frac{1}{5}(x_j-x_i)\exp\left\{-\frac{1}{10}(x_j-x_i)^2\right\}$$
$$= \sum_i\sum_j\left[\beta_i\beta_j + \frac{1}{5}\left[1-\frac{1}{5}(x_j-x_i)^2\right]\gamma_i\gamma_j + \frac{1}{5}(x_j-x_i)(\gamma_i\beta_j - \beta_i\gamma_j)\right]$$
$$\times \exp\left\{-\frac{1}{10}(x_j-x_i)^2\right\}$$
$$= \sum_i\sum_j\left[\beta_i\beta_j + \frac{1}{5}\left[1-\frac{1}{5}(x_j-x_i)^2\right]\gamma_i\gamma_j + \frac{2}{5}(x_j-x_i)\beta_i\gamma_j\right]$$
$$\times \exp\left\{-\frac{1}{10}(x_j-x_i)^2\right\}$$

Solution:

$$\left(\hat{\beta}, \hat{\gamma}\right) = \underset{\beta,\gamma}{\operatorname{argmin}} \; Var\left\{\hat{a} - a\right\}$$

$$\sum_i \beta_i x_i^2 + 2\sum_i \gamma_i x_i = 0$$

$$\sum_i \gamma_i + \sum_i \beta_i x_i = \frac{1}{2}$$

$$\sum_i \beta_i = 0$$

Problem 2 Event RF

$$F_{\underline{X}} : \{ \underline{X}_i ; i=1,\dots, N ; D \subset \mathbb{R}^2 \}$$

$$D : [0,10] \times [0,10]$$

Poisson RF parameter $\lambda \geq 0$

a) Since Poisson RF:

$$N \rightarrow Poi[\lambda \underbrace{|D|}_{100}] = \frac{(100\lambda)^n}{n!} \exp\{-100\lambda\} \; ; n = 0, 1, \dots$$

$$Prob\{N=0\} = \frac{(100\lambda)^0}{0!} \exp\{-100\lambda\} = \exp\{-100\lambda\}$$

$$E\{N\} = 100\lambda$$

$$Prob\{N_\Delta = n_\Delta | N = n\} = \binom{n}{n_\Delta} \left[\frac{|D_\Delta|}{|D|}\right]^{n_\Delta} \left[1 - \frac{|D_\Delta|}{|D|}\right]^{n - n_\Delta}$$

$$= \binom{10}{3} \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^7$$

Cox felt:

$$\{ \underline{X}_i ; i=1,\dots, N | \lambda \} \rightarrow \text{Poisson RF param } \lambda \geq 0$$

$$\lambda \rightarrow Exp\{\mu\} = \frac{1}{\mu} \exp\{-\frac{\lambda}{\mu}\} \; ; \lambda \geq 0$$

$$\Rightarrow E\{\lambda\} = \mu \; ; Var\{\lambda\} = \mu^2$$

b)

$$E\{N\} = E_\lambda\{E_N\{N|\lambda\}\}$$
$$= E_\lambda\{100\lambda\} = \underline{100\mu}$$

$$Var\{N\} = E_\lambda\{Var_N\{N|\lambda\}\} + Var_\lambda\{E_N\{N|\lambda\}\}$$
$$= E_\lambda\{100\lambda\} + Var_\lambda\{100\lambda\}$$
$$= 100\mu + 100^2\mu^2 = \underline{[1+100\mu]100\mu}$$

cente of D

$$S = \underset{x \in \Pi_\Sigma}{Min}\{|c-x|\}$$

c)

$$F(s|\lambda) = Prob\{S<s|\lambda\} = 1 - Prob\{N(sB_0)=0|\lambda\}$$
$$= 1 - \exp\{-\lambda\pi s^2\} \quad ; \quad s \geq 0$$
$$f(s|\lambda) = \frac{d}{ds}F(s|\lambda) = 2\lambda\pi s \exp\{-\lambda\pi s^2\} \quad ; \quad s \geq 0$$
$$f(s) = \int_0^\infty f(s|\lambda) f(\lambda)\, d\lambda$$
$$= \int_0^\infty 2\lambda\pi s \exp\{-\lambda\pi s^2\} \frac{1}{\mu}\exp\{-\frac{\lambda}{\mu}\}\, d\lambda$$
$$= \frac{2\pi s}{\mu}\int_0^\infty \lambda \exp\{-(\pi s^2 + \frac{1}{\mu})\lambda\}\, d\lambda$$

$$\left[\int_0^\infty \lambda e^{-a\lambda}\,d\lambda = \Big|_0^\infty \frac{-1}{a}\left[\lambda + \frac{1}{a}\right]e^{-\lambda a}\right]$$

$$= \frac{2\pi s}{\mu}\Big|_0^\infty \frac{-1}{\left(\pi s^2+\frac{1}{\mu}\right)}\left[\lambda + \frac{1}{\left(\pi s^2+\frac{1}{\mu}\right)}\right]\exp\left\{-\left(\pi s^2+\frac{1}{\mu}\right)\lambda\right\}$$

$$= \frac{2\pi s}{\mu}\left[0 - \frac{-1}{\left(\pi s^2+\frac{1}{\mu}\right)}\frac{1}{\left(\pi s^2+\frac{1}{\mu}\right)}\right]$$

$$= \frac{2\pi s\mu}{(\pi\mu s^2+1)^2} \qquad ; \quad s \geq 0$$

Problem 3 Mosaic RF

$L : \{ L_x ; x \in \mathcal{L}_D \}$ $\quad L_x \in \Omega_L : \{ W, B \}$

$\uparrow$ n nodes

Gibbs model:

$$\mathrm{Prob}\{ L = \ell ; \beta \} = \mathrm{const}_\beta \times \exp\Big\{ \beta \sum_{\langle u,v \rangle} I(\ell_u = \ell_v) \Big\}$$

β $\uparrow$ known

a) Algorithm: single-site McMC M-H

Initiate

ℓ^0 ; $\mathrm{Prob}\{ L = \ell^0 \} > 0$

Iterate $i = 0, 1, \ldots$

$$\left.\begin{array}{l} k \rightsquigarrow \mathrm{Uni}[1, \ldots, n] \\ \ell_k^p \rightsquigarrow q_{ss}(\ell_k \mid \ell^i) \\ \ell^p = (\ell_1^i, \ldots, \ell_k^p, \ldots, \ell_n^i) \end{array}\right\} q(\ell^p \mid \ell^i)$$

$$\ell^{i+1} = \begin{cases} \ell^p & \text{with prob } \alpha(\ell^p \mid \ell^i) \\ \ell^i & \text{else} \end{cases}$$

with $\alpha(\ell^p \mid \ell^i) = \max\Big\{ 1, \frac{\mathrm{Prob}\{L = \ell^p\}}{\mathrm{Prob}\{L = \ell^i\}} \frac{q(\ell^i \mid \ell^p)}{q(\ell^p \mid \ell^i)} \Big\}$

end iterate

then

$$\ell^i \rightsquigarrow \mathrm{Prob}^i\{ L = \ell \} \xrightarrow{i \to \infty} \mathrm{Prob}\{ L = \ell \}$$

Proposals:

(A) $$q_{ss}(l_k | l^i) = q_{ss}(l_k | l_k^i) = \begin{cases} W & \text{if } l_k^i = B \\ B & \text{if } l_k^i = W \end{cases}$$

$$q(l^P | l^i) = \frac{1}{n} \quad \Rightarrow \quad q(l^i | l^P) = \frac{1}{n}$$

$$\alpha(l^P | l^i) = Min\left\{1, \frac{Prob\{L = l^P\}}{Prob\{L = l^i\}} \cdot \frac{1/n}{1/n}\right\}$$

$$\frac{const \times \exp\{\beta \sum_{\substack{\langle u,v \rangle \\ u,v \neq k}} I(l_u^i = l_v^i)\} \exp\{\beta \sum_{\langle k,v \rangle} I(l_k^P = l_v^i)\}}{const \times \exp\{\beta \sum_{\substack{\langle u,v \rangle \\ u,v \neq k}} I(l_u^i = l_v^i)\} \exp\{\beta \sum_{\langle k,v \rangle} I(l_k^i = l_v^i)\}}$$

$$= \exp\left\{\beta \sum_{\langle k,v \rangle} \left[I(l_k^P = l_v^i) - I(l_k^i = l_v^i) \right]\right\}$$

$$I(l_k^P = l_v^i) = [1 - I(l_k^i = l_v^i)]$$

$$= \exp\left\{\beta\, 2\left[2 - \sum_{\langle k,v \rangle} I(l_k^i = l_v^i)\right]\right\} = \alpha_A$$

4 terms - neighb of k.

(B) $$q_{ss}(l_k | l^i) = q_{ss}(l_k) = \begin{cases} W & \text{with prob } 1/2 \\ B & \text{with prob } 1/2 \end{cases}$$

$$q(l^P | l^i) = \frac{1}{n} \cdot \frac{1}{2} \quad \Rightarrow \quad q(l^i | l^P) = \frac{1}{n} \cdot \frac{1}{2}$$

$$\alpha(l^P | l^i) = \cdots = \exp\left\{\beta \sum_{\langle k,v \rangle} \left[I(l_k^P = l_v^i) - I(l_k^i = l_v^i) \right]\right\}$$

NOTE:

$$\alpha(l^P | l^i) = \begin{cases} 1 & \text{with prob } q_{ss}(l_k^i) = \frac{1}{2} \\ \underbrace{\exp\left\{\beta\, 2\left[2 - \sum_{\langle k,v \rangle} I(l_k^i = l_v^i)\right]\right\}}_{= \alpha_A} & \text{else} \end{cases}$$

Compare Algorithm A and B

$$\text{Prob}\{l^i \neq l^{i+1}\} = \text{Prob}\{l^P \neq l^i \cap \text{accept}\}$$

$$= \text{Prob}\{\text{accept} \,/\, l^P \neq l^i\} \, \text{Prob}\{l^P \neq l^i\}$$

Alg A

$$\text{Prob}\{l^i \neq l^{i+1}\} = \alpha_A \cdot 1 = \alpha_A$$

Alg B

$$\text{Prob}\{l^i \neq l^{i+1}\} = \alpha_A \, \frac{1}{2} = \frac{\alpha}{2}$$

Hence:

Alg A has twice the probability of change as Alg B.

For single-site MCMC M-H we expect Alg A to be favorable both for convergence and mixing.