LØSNINGSFORSLAG - EKSAMEN

FAG 75563 ROMLIG STATISTIKK

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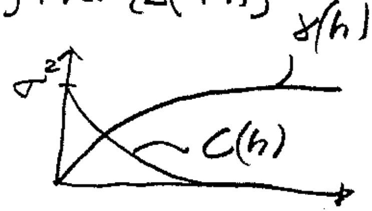
Ref. H. Omre IMF/FIM/NTNU

OPPGAVE 1 KONTINUERLIGE FELT

a) $$\gamma(h) = \frac{1}{2} Var\{Z(x) - Z(x+h)\} = \frac{1}{2}\Big[Var\{Z(x)\} + Var\{Z(x+h)\}$$

$$- 2Cov\{Z(x), Z(x+h)\} = \underline{\sigma^2 - C(h)}$$



Husk at:

$$C(h) = Cov\{Z(x), Z(x+h)\} = E\{Z(x) Z(x+h)\} - E\{Z(x)\}E\{Z(x+h)\}$$

$$= E\{Z(x)Z(x+h)\} - a^2$$

$$\gamma(h) = \frac{1}{2} Var\{Z(x) - Z(x+h)\}$$

$$= \frac{1}{2} E\{[Z(x) - Z(x+h)]^2\} - \Big[E\{Z(x) - Z(x+h)\}\Big]^2$$

$$= \frac{1}{2} E\{(Z(x) - Z(x+h))^2\}$$

Det er åpenbart at de oppgitte estimatorene er rimelige.

$$E\{\hat{C}(h)\} = \frac{1}{N_h} \sum E\{Z(x_i)Z(x_j)\} - \frac{1}{n^2} E\Big\{\Big(\sum_i Z(x_i)\Big)^2\Big\}$$

$$= C(h) + a^2 - \frac{1}{n^2} \sum_i^n \sum_j^n \big[C(x_i - x_j) + a^2\big]$$

$$= C(h) - \frac{1}{n^2} \sum_i \sum_j C(x_i - x_j)$$

– altså ikke forventningsrett

$$E\{\hat{\gamma}(h)\} = \frac{1}{2N_h}\sum_i E\{(Z(x_i) - Z(x_j))^2\}$$

$$= \frac{1}{2N_h}\sum_i \left[Var\{Z(x_i) - Z(x_j)\} + \left[E\{Z(x_i) - Z(x_j)\}\right]^2\right]$$

$$= \frac{1}{2} Var\{Z(x_i) - Z(x_j)\} = \gamma(h)$$

– altså forventningsrett.

Det vil altså være bedre å bruke variogrammet som parameter i modellen fordi det kan estimeres forventningsrett.

b) Positiv definitt:

$$\sum_i^n \sum_j^n \alpha_i \alpha_j C(|x_i - x_j|) \geq 0 \quad \begin{cases} \text{alle } \alpha_i \\ \text{alle konf. } x_i \\ \text{alle } n. \end{cases}$$

En ønsker at alle lineærkombinasjoner $\sum \alpha_i Z(x_i)$ skal ha ikke-negativ varians

$C_1(\cdot)$ pos def $\Rightarrow \sum\sum \alpha_i \alpha_j C_1(|x_i - x_j|) \geq 0$ alle.

$C_2(\cdot)$ pos. def $\Rightarrow \sum\sum \beta_i \beta_j C_2(|x_i - x_j|) \geq 0$

Åpenbart at summen må være pos. def.

$$\sum_i^n \sum_j^n \nu_i \nu_j (C_1(x_i - x_j) + C_2(x_i - x_j)) \geq 0$$

$$\begin{cases} \text{alle } \nu_i \\ \text{alle } x_i \\ \text{alle } n \end{cases}$$

c) Betrakt

$$Z^*(x_0) = \sum_{i=1}^{n} \alpha_i Z(x_i)$$

Beste lineære forventn. rette prediktor under kvadratisk tap er definert ved:

$$\underline{\alpha}^* = \underset{\underline{\alpha}}{\operatorname{argmin}} \left\{ Var\left\{ Z(x_0) - \sum_{i=1}^{n} \alpha_i Z(x_i) \right\} \right\}$$

beskranket av

$$E\left\{ Z(x_0) - \sum_{i=1}^{n} \alpha_i Z(x_i) \right\} = 0$$

Forventningsretthetskravet er:

$$E\{Z(x_0)\} = E\left\{ \sum_i \alpha_i Z(x_i) \right\}$$

$$\Downarrow$$

$$a_0 + \sum_{\ell=1}^{L} a_\ell g^\ell(x_0) = \sum_i \alpha_i \left[a_0 + \sum_{\ell=1}^{L} a_\ell g^\ell(x_i) \right]$$

$$\Downarrow$$

$$\sum_{\ell=0}^{L} a_\ell g^\ell(x_0) = \sum_{\ell=0}^{L} a_\ell \sum_{i=1}^{n} \alpha_i g^\ell(x_i) \quad ; \text{ med } g^0(x) = 1$$

$$\Downarrow$$

$$g^\ell(x_0) = \sum_{i=1}^{n} \alpha_i g^\ell(x_i) \quad ; \quad \ell = 0, \ldots, L$$

Minimeringskriteriet:

$$Var\left\{ Z(x_0) - \sum_i \alpha_i Z(x_i) \right\}$$

$$= \sigma^2 - 2 \sum_i \alpha_i C(|x_0 - x_i|) + \sum_i \sum_j \alpha_i \alpha_j C(|x_i - x_j|,$$

d) Betrakt

$$\bar{Z}_D^* = \sum_i \beta_i Z(x_i)$$

Beste lineære forventningsrette prediktor under kvadratisk tap er definert ved:

$$\beta^* = \underset{\beta}{\operatorname{argmin}} \left\{ Var\left\{ \bar{Z}_D - \sum_{i=1}^{n} \beta_i Z(x_i) \right\} \right\}$$

beskranket av

$$E\left\{ \bar{Z}_D - \sum_{i=1}^{n} \beta_i Z(x_i) \right\} = 0$$

Forventningsretthetskravet er:

$$E\{\bar{Z}_D\} = E\left\{ \sum_{i=1}^{n} \beta_i Z(x_i) \right\}$$

$$\Downarrow$$

$$\frac{1}{|D|}\int_D \sum_{l=0}^{L} a_l g^l(u)\,du = \sum_i \beta_i \sum_{l=0}^{L} a_l g^l(x_i) \;;\; g^0(x) = 1$$

$$\Downarrow$$

$$\frac{1}{|D|}\int_D g^l(u)\,du = \sum_i \beta_i g^l(x_i) \;;\; l = 0, \dots, L$$

Minimeringskriteriet:

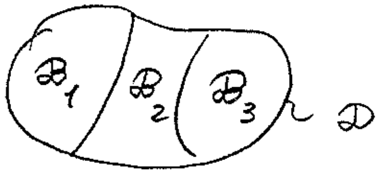
$$Var\left\{ \bar{Z}_D - \sum \beta_i Z(x_i) \right\}$$

$$= \frac{1}{|D|^2}\iint_{DD} C(u-v)\,du\,dv - \frac{2}{|D|}\sum_i \beta_i \int_D C(x_i - u)\,du$$

$$+ \sum_i \sum_j \beta_i \beta_j C(x_i - x_j)$$

OPPGAVE 2 HENDELSESFELT

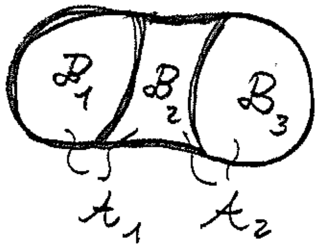
a)



dvs $N(\mathcal{B}_1), N(\mathcal{B}_2), N(\mathcal{B}_3)$ uavh. tilf. variable:

$$\text{Prob}\{N(\mathcal{B}_1)=n_1, N(\mathcal{B}_2)=n_2, N(\mathcal{B}_3)=n_3\}$$
$$= \prod_{i=1}^{3} \text{Prob}\{N(\mathcal{B}_i)=n_i\} = \prod_{i=1}^{3} \frac{\lambda^{n_i}|\mathcal{B}|^{n_i}}{n_i!} e^{-\lambda|\mathcal{B}|}$$

b)



dvs $N(\mathcal{A}_1), N(\mathcal{A}_2)$ avh. via $N(\mathcal{B}_2)$.

$$\text{Prob}\{N(\mathcal{A}_1)=m_1, N(\mathcal{A}_2)=m_2\}$$
$$= \sum_{i=1}^{M} \text{Prob}\{N(\mathcal{A}_1)=m_1, N(\mathcal{A}_2)=m_2, N(\mathcal{B}_2)=i\}$$

med $M = \min\{m_1, m_2)$

$$= \sum_{i=1}^{M} \text{Prob}\{N(\mathcal{B}_1)+N(\mathcal{B}_2)=m_1, N(\mathcal{B}_2)+N(\mathcal{B}_3)=m_2 \mid N(\mathcal{B}_2)=i\} \cdot \text{Prob}\{N(\mathcal{B}_2)=i\}$$
$$= \sum_{i=1}^{M} \text{Prob}\{N(\mathcal{B}_1)=m_1-i, N(\mathcal{B}_3)=m_2-i\} \cdot \text{Prob}\{N(\mathcal{B}_2)=i\}$$
$$= \sum_{i=1}^{M} \frac{\lambda^{m_1-i}|\mathcal{B}_1|^{m_1-i}}{(m_1-i)!} \frac{\lambda^{m_2-i}|\mathcal{B}_3|^{m_2-i}}{(m_2-i)!} \frac{\lambda^{i}|\mathcal{B}_2|^{i}}{i!} e^{-\lambda|\mathcal{D}|}$$

med $M = \min\{m_1, m_2\}$

c) En har.

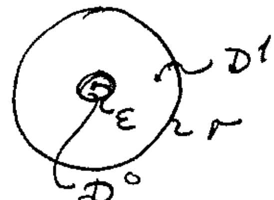
$$F_R(r) = \text{Prob}\{R(x_0) < r\} = 1 - \text{Prob}\{R(x_0) > r\}$$

$$= 1 - \frac{(\lambda \pi r^2)^0}{0!} e^{-\lambda \pi r^2} = 1 - e^{-\lambda \pi r^2}$$

dvs

$$f_R(r) = \frac{dF_R(r)}{dr} = \underline{2\pi r e^{-\lambda \pi r^2}}$$

Gitt punkt i x_0, definer



$\mathcal{D} = \mathcal{D}^0 \cup \mathcal{D}^1$

$$\lim_{\varepsilon \to 0} \frac{\text{Prob}\{(\text{ikke pkt i } \mathcal{D}^1) \cap (\text{pkt i } \mathcal{D}^0)\}}{\text{Prob}\{\text{pkt i } \mathcal{D}^0\}}$$

$$= \lim_{\varepsilon \to 0} \frac{\text{Prob}\{(\text{ikke pkt i } \mathcal{D}^1)\}\,\text{Prob}\{\text{pkt i } \mathcal{D}^0\}}{\text{Prob}\{\text{pkt i } \mathcal{D}^0\}}$$

$$= \lim_{\varepsilon \to 0} \text{Prob}\{\text{ikke pkt i } \mathcal{D}^1\}$$

$$= \text{Prob}\{\text{ikke pkt i } \mathcal{D}\}$$

$$F_R(r) = 1 - \text{Prob}\{R(x_0) > r\} = 1 - e^{-\lambda \pi r^2}$$

dvs

$$f_R(r) = \frac{dF_R(r)}{dr} = 2\pi r\, e^{-\lambda \pi r^2}$$

De to pdf'ene er altså identiske. Dette er en unik egenskap ved Poisson punktfelt.

OPPGAVE 3 MOSAIKKFELT

a) Det er åpenbart at

$$Q_{ll'} = Q_{l'l}$$

Videre

$$l_{ij} l_{kl} = l'_{ij} l'_{kl}$$ for alle ij og kl som ikke inkluderer den node som endres dvs (m,n)

dvs

$$\alpha_{ll'} = \min\{1, \exp\{-\beta(l'_{mn} - l_{mn}) \cdot (l_{m+1,n} + l_{m-1,n} + l_{m,n+1} + l_{m,n-1})\}\}$$

Dvs. at kun de fire nærmeste naboene til den node som oppdateres er involvert i uttrykket for aksept-sannsynligheten. Det gjør algoritmen mye mer effektiv.