Løsningsskisse for eksamen i 75563 Romlig statistikk,
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Oppg. 1 a)

Krav: $\rho(x,x) = 1$ fordi $\rho(x,x) = Corr[Z(x), Z(x)] = 1$

$\rho(x,x')$ må være positivt definitt, dvs.

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \rho(x_i, x_j) \quad \forall\ n \text{ og alle } c_i;\ i=1,\dots,n$$

Dette kravet følger av at $Var\left[\sum_{i=1}^{n} c_i Z(x_i)\right] \geq 0$.

Har at:

$$Var\left[\sum_{i=1}^{n} c_i' Z(x_i)\right] = Cov\left[\sum_{i=1}^{n} c_i' Z(x_i), \sum_{j=1}^{n} c_j' Z(x_j)\right]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i' c_j' Cov\left[Z(x_i), Z(x_j)\right]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i' c_j' \cdot \frac{Corr\left[Z(x_i), Z(x_j)\right]}{\sqrt{Var\left[Z(x_i)\right] \cdot Var\left[Z(x_j)\right]}}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \cdot \rho(x_i, x_j) \quad \text{der} \quad c_i = \frac{c_i'}{\sqrt{Var(Z(x_i))}}$$

b) $\hat{Z}(x_0) = \alpha_0 + \sum_{i=1}^{n} \alpha_i Z(x_i)$

Forventningsretthet:

$$E\left[\hat{Z}(x_0) - Z(x_0)\right] = \mu(x_0) - \alpha_0 - \sum_{i=1}^{n} \alpha_i \mu(x_i)$$

$$= 0$$

$$\Rightarrow \quad \alpha_0 = \mu(x_0) - \sum_{i=1}^{n} \alpha_i \mu(x_i)$$

dvs. $$\hat{Z}(x_0) = \mu(x_0) + \sum_{i=1}^{n} \alpha_i \left(Z(x_i) - \mu(x_i) \right)$$

$$= \mu(x_0) + \alpha^T (\underset{\sim}{Z} - \underset{\sim}{\mu})$$

der $\underset{\sim}{Z} = (Z(x_1), \ldots, Z(x_n))^T$

$\underset{\sim}{\alpha} = (\alpha_1, \ldots, \alpha_n)^T$

$\underset{\sim}{\mu} = (\mu(x_1), \ldots, \mu(x_n))^T$

$$\mathrm{Var}\left[\hat{Z}(x_0) - Z(x_0)\right] = \mathrm{Var}\left[\mu(x_0) + \underset{\sim}{\alpha}^T (\underset{\sim}{Z} - \underset{\sim}{\mu}) - Z(x_0)\right]$$

$$= \mathrm{Var}\left[\underset{\sim}{\alpha}^T \underset{\sim}{Z} - Z(x_0)\right] = \mathrm{Var}\left[[\underset{\sim}{\alpha}^T \; -1] \begin{bmatrix} \underset{\sim}{Z} \\ Z(x_0) \end{bmatrix} \right]$$

$$= [\underset{\sim}{\alpha}^T \; -1] \cdot \begin{bmatrix} \Sigma & b \\ \underset{\sim}{b}^T & \sigma^2(x_0) \end{bmatrix} \begin{bmatrix} \underset{\sim}{\alpha} \\ -1 \end{bmatrix}$$

der $\quad \Sigma \in \mathbb{R}^{n \times n}$ med $\Sigma_{ij} = \sigma(x_i)\sigma(x_j)\rho(x_i, x_j)$

$\quad\quad \underset{\sim}{b} \in \mathbb{R}^n$ med $b_i = \sigma(x_i)\sigma(x_0)\rho(x_i, x_0)$

dvs.

$$\mathrm{Var}\left[\hat{Z}(x_0) - Z(x_0)\right] = \underset{\sim}{\alpha}^T \Sigma \underset{\sim}{\alpha} - \underset{\sim}{\alpha}^T \underset{\sim}{b} - \underset{\sim}{b}^T \underset{\sim}{\alpha} + \sigma^2(x_0)$$

$$= \underset{\sim}{\alpha}^T \Sigma \underset{\sim}{\alpha} - 2\underset{\sim}{\alpha}^T \underset{\sim}{b} + \sigma^2(x_0)$$

Gradient m.h.p. $\underset{\sim}{\alpha}$:

$$\nabla_{\underset{\sim}{\alpha}} \left(\underset{\sim}{\alpha}^T \Sigma \underset{\sim}{\alpha} - 2\underset{\sim}{\alpha}^T \underset{\sim}{b} + \sigma^2(x_0) \right)$$

$$= 2\Sigma\underset{\sim}{\alpha} - 2\underset{\sim}{b} = 0$$

$$\Rightarrow \underset{\sim}{\alpha} = \Sigma^{-1}\underset{\sim}{b}$$

dvs. $\quad \underset{\sim}{\alpha} = \Sigma^{-1}\underset{\sim}{b}$

$$\underline{\underline{\alpha_0 = \mu(x_0) - \underset{\sim}{\alpha}^T \underset{\sim}{\mu}}}$$

Oppgave 2

a) $\{x \notin B\} = \{$ingen punkter i området $b(x;r)\}$

Benytter at antall punkter i et område A er Poissonfordelt med parameter $\lambda \cdot |A|$.

Spesielt:

$$Pr\{N(b(x;r)) = k\} = \frac{(\lambda \pi r^2)^k}{k!} e^{-\lambda \pi r^2} \quad ; \; k = 0, 1, \ldots$$

$$Pr\{x \notin B\} = Pr\{N(b(x;r)) = 0\} = \underline{\underline{e^{-\lambda \pi r^2}}}$$

b) $$|B| = \int_W I(x \in B)\, dx$$

$$E[|B|] = \int_W E[I(x \in B)]\, dx = \int_W Pr\{x \in B\}\, dx$$

$$= \int_W \left(1 - e^{-\lambda \pi r^2}\right) dx = \underline{\underline{\left(1 - e^{-\lambda \pi r^2}\right) \cdot |W|}}$$

Definerer estimator, $\hat{\lambda}$, ved å kreve:

$$|B| = \left(1 - e^{-\hat{\lambda} \pi r^2}\right) \cdot |W|$$

$$\Rightarrow \underline{\underline{\hat{\lambda} = -\frac{1}{\pi r^2} \cdot \ln\left(1 - \frac{|B|}{|W|}\right)}}$$

Z_{ij}: posisjon til senter av piksel (i,j).

c) $$\Pr\{Y_{ij}=1\} \underset{\uparrow \atop TS}{=} \Pr\{Y_{ij}=1 \mid Z_{ij}\in B\}\cdot \Pr\{Z_{ij}\in B\}$$

$$+ \Pr\{Y_{ij}=1 \mid Z_{ij}\notin B\}\cdot \Pr\{Z_{ij}\notin B\}$$

$$= p\cdot\left(1-e^{-\lambda\pi r^2}\right) + (1-p)\, e^{-\lambda\pi r^2}$$

$$= p + (1-2p)\, e^{-\lambda\pi r^2} = E[Y_{ij}]$$

$$E\left[\sum_{i=1}^{n}\sum_{j=1}^{n} Y_{ij}\right] = n_1\cdot n_2\cdot\left(p+(1-2p)\,e^{-\lambda\pi r^2}\right)$$

Definerer estimatoren, $\tilde{\lambda}$, ved å kreve:

$$\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} Y_{ij} = n_1\cdot n_2\left(p+(1-2p)\,e^{-\tilde{\lambda}\pi r^2}\right)$$

$$\Rightarrow \underline{\underline{\tilde{\lambda} = -\frac{1}{\pi r^2}\cdot\left[\ln\left(\frac{1}{n_1 n_2}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} Y_{ij} - p\right) - \ln(1-2p)\right]}}$$

$p=\frac{1}{2} \Rightarrow \tilde{\lambda}$ blir udefinert fordi $1-2p=0$

Dette er rimelig fordi med $p=\frac{1}{2}$ vil fordelingen til Y_{ij} ikke være avhengig av λ.

$\Pr\{Y_{ij}=1\}=\frac{1}{2}$ uansett hva λ er.

Oppg. 3

a) Nummerer mulige tilstander for $\underset{\sim}{X}$ fra 1 til 2^n. La p_i være sanns. til tilstand nr. i. La videre

$$q_i = \sum_{j=1}^{i} p_i \quad \text{(kumulativ fordeling)}$$

Simuleringsalgoritme:

Trekk $u \sim \text{Unif}[0,1]$

Velg tilstand nr. i slik at

$$q_{i-1} \leq u < q_i$$

Dette er i praksis ikke gjennomførbart fordi man må regne ut 2^n p_i-er og q_i-er. For $n = 100^2$ får en $2^n \approx 10^{6.931}$, dvs. ikke gjennomførbart på dagens datamaskiner.

Gibbs-sampler (spesialtilfelle av Metropolis-Hastings) for Ising-modell benytter

$$q(x \to x') = \begin{cases} 0 & \text{hvis } x \text{ og } x' \text{ avviker i mer enn ein piksel} \\ \Pr\{X_i = x_i' \mid X_{-i} = x_{-i}\} & \text{dersom } x \text{ og } x' \text{ avviker i piksel } i. \end{cases}$$

$\alpha(x \to x') = 1$ alltid for Gibbs-sampler.

b)

$$l(\beta) = P_r\{X = x\} = c(\beta) \cdot e^{\beta \sum_{i \sim j} I(x_i = x_j)}$$

$$\text{der } c(\beta) = \frac{1}{\sum_{x'} e^{\beta \sum_{i \sim j} I(x_i' = x_j')}}$$

antall ledd i summen er 2^n. Konstanten $c(\beta)$ lar seg derfor (i praksis) ikke beregne og dermed kan en heller ikke maksimere $l(\beta)$

Betinget fordeling i Ising-modell:

$$\Pr\{X_i = x_i \mid X_{-i} = x_{-i}\} = c(x_{-i}) \cdot e^{+\beta \sum_{j \in \partial i} I(x_i = x_j)}$$

der $$c(x_{-i}) = \frac{1}{e^{+\beta \sum_{j\in\partial i} I(0 = x_j)} + e^{+\beta \sum_{j \in \partial i} I(1 = x_j)}}$$

$$pl(\beta) = \prod_{i=1}^{n} \left[\Pr_\beta \{X_i = x_i \mid X_{-i} = x_{-i}\} \right]$$

$$= \prod_{i=1}^{n} \left[\frac{e^{+\beta \sum_{j \in \partial i} I(x_i = x_j)}}{e^{+\beta \sum_{j \in \partial i} I(x_j = 0)} + e^{+\beta \sum_{j \in \partial i} I(x_j = 1)}} \right]$$

$$\ln[pl(\beta)] = \sum_{i=1}^{n} \Big[\beta \sum_{j \in \partial i} I(x_i = x_j) - \ln\Big[e^{\beta \sum_{j \in \partial i} I(x_j = 0)} + e^{\beta \sum_{j\in\partial i} I(x_j = 1)} \Big] \Big]$$

$$\frac{d}{d\beta} \ln\left[pl(\beta) \right] = \sum_{i=1}^{n} \Bigg[\sum_{j \in \partial i} I(x_i = x_j)$$

$$- \frac{\left[\sum_{j\in\partial i} I(x_j=0)\right] \cdot e^{\beta \sum_{j\in\partial i} I(x_j=0)} + \left[\sum_{j\in\partial i} I(x_j=1)\right] e^{\beta \sum_{j\in\partial i} I(x_j=}}{e^{\beta \sum_{j\in\partial i} I(x_j=0)} + e^{\beta \sum_{j\in\partial i} I(x_j=1)}}$$

$$= 0$$