LØSNINGSFORSLAG
Fag 75563 Romlig Statistikk
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Ref: H. Omre IMF/FIM/NTNU

Oppgave 1 Kontinuerlige felt

$\{R(x); x \in \mathbb{R}^1\}$ $\quad E\{R(x)\} = 0$

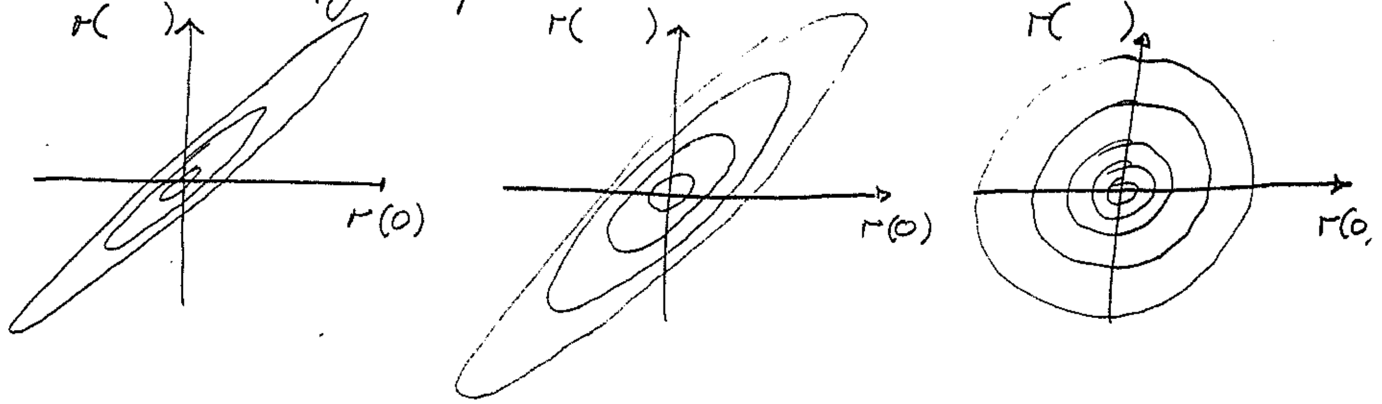
$\quad Var\{R(x)\} = 1$

$\{R'(x) = \frac{dR(x)}{dx}; x \in \mathbb{R}^1\}$ $\quad Cov\{R(x), R(x+h)\} = C(h) = e^{-|h|^2}$

a) Krav til Gaussiske felt

$$[R(x_1), \ldots, R(x_k)] \sim Gauss[\;,\;]$$

alle konfigurasjoner $(x_1, \ldots, x_k)$; alle $k \in \mathbb{N}^+$

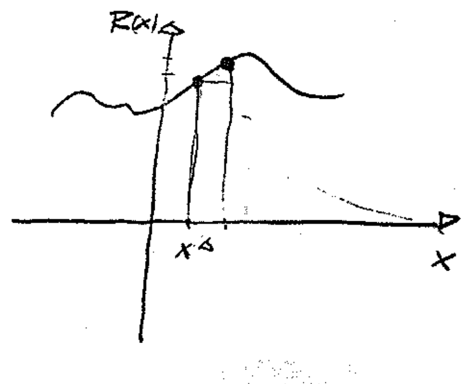


b) Krav til eksistens av $\{R'(x); x \in \mathbb{R}^1\}$

$$\left.\frac{d^2C(h)}{dh}\right|_{h=0} \text{ - eksisterer}$$

$$Cov\{R'(x), R(x+h)\} = Cov\{\lim_{\Delta \to 0} \frac{R(x+\Delta) - R(x)}{\Delta}, R(x+h)\}$$

$$= \lim_{\Delta \to 0} \frac{1}{\Delta}\{C(h-\Delta) - C(h)\}$$

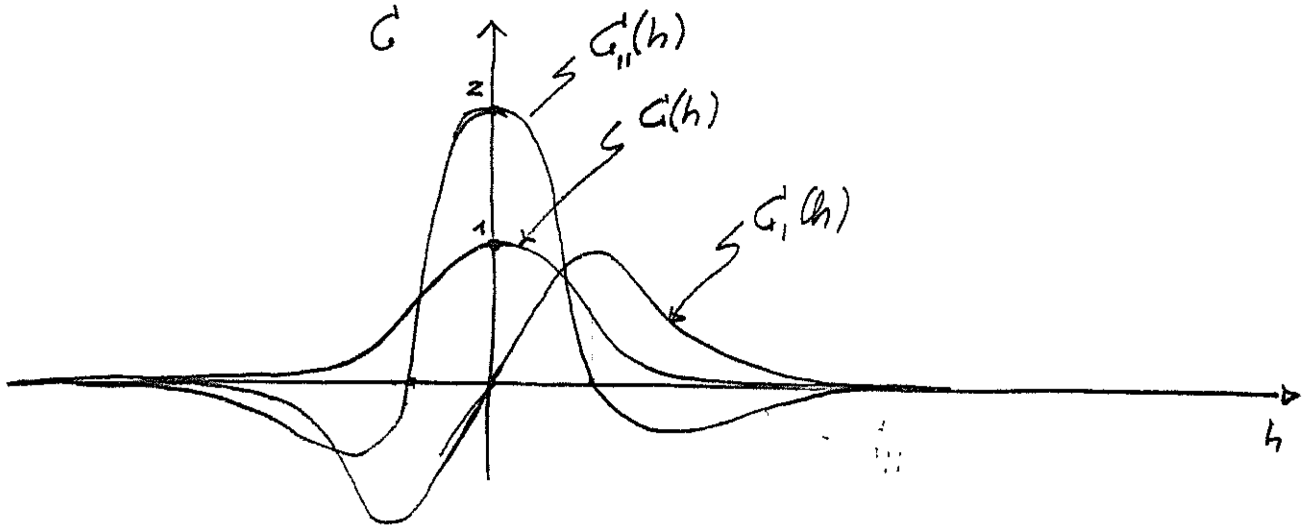


$$= -\frac{dC(h)}{dh}$$

$$= 2he^{-h^2} = C_1(h)$$

$$\mathrm{Cov}\{R'(x), R'(x)\} = -\frac{d^2 G(h)}{dh^2}$$

$$= -\left[-2e^{-|h|^2} + 4h^2 e^{-h^2}\right] = 2e^{-h^2}[1-2h] = G_{11}(h)$$



Merk at $G_1(0) = \mathrm{Cov}\{R'(x), R(x)\} = 0$, dvs at de er ukorrelerte, og altså uavhengige i det Gaussiske tilfellet.

c) Bruk prediktoren:

$$\hat{R}_\alpha(x) = \alpha[R'(0) - \underbrace{E\{R'(0)\}}_{0}] + \underbrace{E\{R(x)\}}_{0}$$

$$= \alpha R'(0)$$

Bestem:

$$\alpha^* = \arg\min_\alpha \mathrm{Var}\{R(x) - \hat{R}_\alpha(0)\}$$

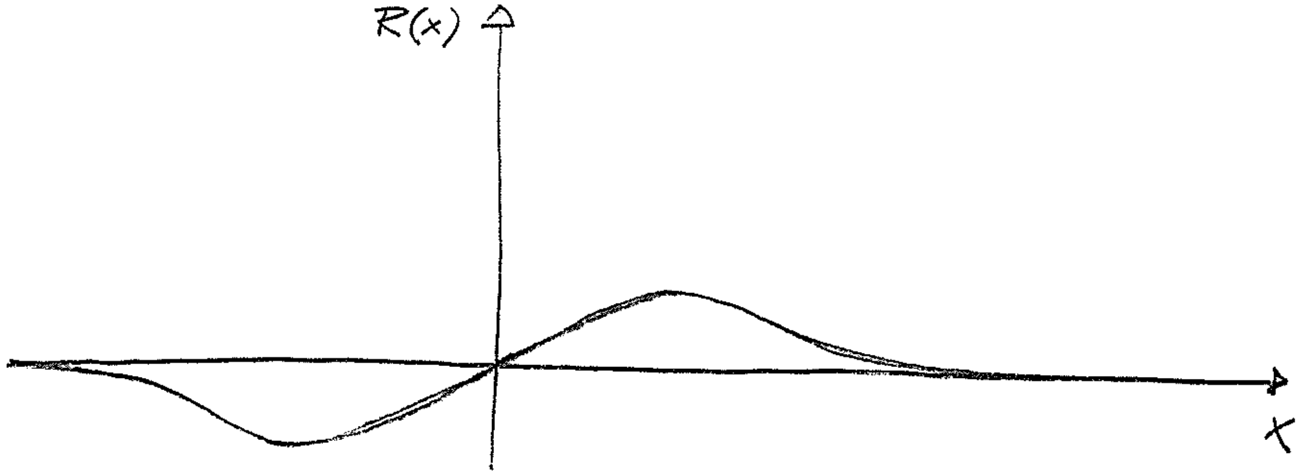
$$= \arg\min_\alpha \mathrm{Var}\{R(x) - \alpha R'(0)\}$$

$$= \arg\min_\alpha \{G(0) - 2\alpha G_1(x) + \alpha^2 G_{11}(0)\}$$

$$\frac{d}{d\alpha} = 0 \Rightarrow = \frac{G_1(x)}{G_{11}(0)} = \frac{1}{2} G_1(x)$$

Prediktoren blir:

$$\hat{R}_{\alpha^*}(x) = \frac{1}{2}\, C_1(x) \cdot 0.5 = \frac{1}{2} x e^{-x^2}$$



MERK:

- den deriverte i $x=0$ er lik 0.5 som observert
- prediksjonen i $x=0$ er lik $E\{R(x)\}=0$ fordi $R'(0), R(0)$ er ukorrelert.
- prediksjonen går mot $E\{R(x)\}=0$ når $x \to \infty$ fordi da avtar korrelasjonen

Oppgave 2 Hendelsesfelt

Poisson punktfelt

$$\text{Prob}\{N(\mathcal{B})=k\} = \frac{[\lambda|\mathcal{B}|]^k}{k!} e^{-\lambda|\mathcal{B}|}$$

a) $\text{Prob}\{N(\mathcal{B}_1)=2\} = \frac{1}{2}\lambda^2|\mathcal{B}_1|^2 e^{-\lambda|\mathcal{B}_1|}$

$$\text{Prob}\{N(\mathcal{B}_2)=k / N(\mathcal{B}_1)=2\}$$

$$= \frac{\text{Prob}\{N(\mathcal{B}_2)=k\}\cdot\text{Prob}\{N(\mathcal{B}_1)=2\}}{\text{Prob}\{N(\mathcal{B}_1)=2\}}$$

$$= \text{Prob}\{N(\mathcal{B}_2)=k\} = \frac{[\lambda|\mathcal{B}_2|]^k}{k!} e^{-\lambda|\mathcal{B}_2|}$$

7

b) $F_2(r_2) = \text{Prob}\{R_{(2)} \leq r_2\} = 1 - \text{Prob}\{N(r_1\mathcal{B}_0) \leq 1\}$

$$= 1 - e^{-\lambda\pi r_2^2} - \lambda\pi r_2^2 e^{-\lambda\pi r_2^2}$$

$$f_2(r_2) = \frac{dF_2(r_2)}{dr_2} = \underline{\underline{2\lambda^2\pi^2 r_2^3 e^{-\lambda\pi r_2^2}}}$$

$$F(r_1, r_2) = \text{Prob}\{R_{(1)} \leq r_1 \wedge R_{(2)} \leq r_2\}$$

$$f(r_1, r_2) = \frac{d^2F(r_1, r_2)}{dr_1\, dr_2}$$

$$f(r_1, r_2) = f(r_2|r_1)\, f(r_1)$$

$$f(r_1) = 2\lambda\pi r_1 e^{-\lambda\pi r_1^2}$$

$$F(r_2|r_1) = 1 - \text{Prob}\{N(\circledcirc)=0\} = 1 - e^{-\lambda\pi(r_2^2-r_1^2)}$$

$$f(r_2|r_1) = \frac{dF(r_2|r_1)}{dr_2} = 2\lambda\pi r_2 e^{-\lambda\pi(r_2^2-r_1^2)}$$

$$f(r_1, r_2) = \begin{cases} (2\lambda\pi)^2 r_1 r_2 e^{-\lambda\pi r_2^2} & r_1 < r_2 \\ 0 & \text{else} \end{cases}$$

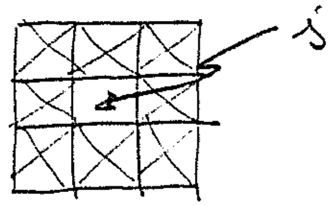
$$f(r_2) = \int_0^{r_2} f(r_2, r_1)\, dr_1 = (2\lambda\pi)^2 r_2 e^{-\lambda\pi r_2^2} \int_0^{r_2} r_1\, dr_1$$

$$= \underline{2\lambda^2\pi^2 r_2^3 e^{-\lambda\pi r_2^2}}$$

Oppgave 3 Mosaic felt

$L = \{L_x ; x \in \mathcal{L}_D\}$; $L_x \in \{1, \dots, K\}$; $\forall x \in \mathcal{L}_D$

a) Naboskap $\mathcal{N}_s$:



Markovantakelse:

$$\text{Prob}\{L_y = l_y \mid L_x = l_x ; x \in \mathcal{L}_D ; x \neq y\}$$
$$= \text{Prob}\{L_y = l_y \mid L_x = l_x ; x \in \mathcal{N}_y\}$$

Definer største clique:

naboskap clique

Gibbs felt:

$$\text{Prob}\{L = l\} = \text{cons} \times \exp\{\sum_{c \in C} u_c(l)\}$$

med

$u_c(l)$ - funksjon av de l som tilhører cliquen c bare

C - mengden av alle cliquer i $\mathcal{L}_D$

Hammersley-Clifford teorem:

Det eksisterer et Markov felt hvis og bare hvis et Gibbs felt eksisterer.

Det betyr at et lokalt definert Markov felt alltid har et tilsvarende globalt definert Gibbs felt.