LØSNINGSFORSLAG

SIF5064 ROMLIG STATISTIKK

10.05.2001

H. Omre

OPPGAVE 1 KONTINUERLIGE FELT

$$\{R(x); x \in \mathbb{R}^n\} \rightarrow \begin{array}{l} E\{R(x)\} = \mu \\ Var\{R(x) - R(x+h)\} = 2 \cdot \gamma(h) \end{array}$$

a) Annen ordens stasjonært

$$\Downarrow$$

$$E\{R(x)\} = \mu \quad \text{uavh. av } x$$

$$Var\{R(x)\} = \sigma^2 < \infty$$

$$Cov\{R(x), R(x+h)\} = C(h) \quad - \text{uavh. av } x$$

pos. def. funksjon

herav

$$\gamma(h) = \frac{1}{2} Var\{R(x) - R(x+h)\}$$

$$= \frac{1}{2}\left[Var\{R(x)\} + Var\{R(x+h)\} - 2\,Cov\{R(x), R(x+h)\}\right]$$

$$= \sigma^2 - C(h)$$

b) IRF-1

$[R(x) - R(x+h)]$ – annen ordens stasjonært $\forall x, \forall h$

$$\Downarrow$$

$$E\{R(x)\} = \mu \quad \text{uavh. av } x$$

$$Var\{R(x) - R(x+h)\} = 2 \cdot \gamma(h) < \infty$$

$[-\gamma(h)]$ må være betinget pos. def. funksjon:

$$-\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j \gamma(x_i - x_j) \geq 0$$

for $\forall \alpha = (\alpha_1, \ldots, \alpha_n)$ slik at $\sum_{i=1}^{n} \alpha_i = 0$

$\forall$ konfigurasjoner $(x_1, \ldots, x_n)$

$\forall n$

I prediksjon ønsker en å predikere $Z(x_0)$ basert på observasjonene $\{Z(x_1), \ldots, Z(x_n)\}$. Lineær, forventnings-rett minste kvadraters estimator er:

$$\hat{Z}(x_0) = \sum_{i=1}^{n} \hat{\alpha}_i Z(x_i)$$

hvor

$$\hat{\alpha} = \underset{\alpha}{\operatorname{argmin}} \left\{ Var\left\{ Z(x_0) - \sum_{i=1}^{n} \alpha_i Z(x_i) \right\} \right\}$$

$$E\left\{ Z(x_0) - \sum_{i=1}^{n} \alpha_i Z(x_i) \right\} = 0$$

herav

Prediksjonsvarians

$$Var\left\{ \sum_{i=0}^{n} \beta_i Z(x_i) \right\} \longrightarrow \begin{cases} \beta_0 = 1 \\ \beta_i = -\alpha_i \quad i = 1, n \end{cases}$$

hvor $\sum_{i=0}^{n} \beta_i = 0$.

Prediksjonsvariansen omhandler kun lineær-kombinasjoner hvor $\sum_{i=0}^{n} \beta_i = 0$, dvs kontraster.

c) IRF-1 med $\gamma(h) = \sigma|h|^{\nu}$ $\sigma > 0,\ 0 \le \nu < 2$

'Affine similar'

$\Downarrow$

$$\{R(x);\ x \in \mathbb{R}^n\} \rightarrow \begin{array}{l} E\{R(x)\} = \mu \\ Var\{R(x) - R(x+h)\} = 2 \cdot \gamma(h) \end{array}$$

uavh. av h!

$$\{R(\xi x);\ x \in \mathbb{R}^n\} \rightarrow \begin{array}{l} E\{R(x)\} = \mu \\ Var\{R(x) - R(x+h)\} = 2 \cdot \omega_{\xi} \cdot \gamma(h) \end{array}$$

dvs reskalering av referanserommet endrer kun skala, men ikke 'variogram struktur' i feltet.

Betrakt

. opprinnelig . reskalert ξ

$$Var\{R(x) - R(x+h)\}$$
$$= 2 \cdot \gamma(h)$$
$$= 2 \cdot \sigma|h|^{\nu}$$

$$Var\{R(\tfrac{1}{\xi}x) - R(\tfrac{1}{\xi}(x+h))\}$$
$$= 2 \cdot \gamma(\tfrac{1}{\xi}h)$$
$$= 2 \cdot \sigma|\tfrac{1}{\xi}h|^{\nu}$$
$$= 2\sigma \cdot \underbrace{\frac{1}{\xi^{\nu}}}_{\omega_{\xi}}|h|^{\nu}$$

OPPGAVE 2   HENDELSES FELT

Stasjonært punktfelt:

$$\lambda = \frac{E\{N(D)\}}{|D|}$$

$$K(r) = E\{N(rB_0)\}/\lambda$$

$\{x_1, \ldots, x_n \in B\}$

a) Poisson punktfelt.

$$\hat{\lambda} = \frac{N}{|B|} \longrightarrow E\{\hat{\lambda}\} = \frac{E\{N\}}{|B|} = \frac{\lambda|B|}{|B|} = \lambda \text{ - forv. rett}$$

$$Var\{\hat{\lambda}\} = \frac{Var\{N\}}{|B|^2} = \frac{\lambda|B|}{|B|^2} = \frac{\lambda}{|B|}$$

$$\hat{K}(r) = \pi r^2 \longrightarrow K(r) \xrightarrow{\text{pr. def}} \frac{\lambda\pi r^2}{\lambda} = \pi r^2$$

Forventningsrett / Varians 0!!

b) Vilkårlig punktfelt.

$$\hat{\lambda} = \frac{N}{|B|} \longrightarrow E\{\hat{\lambda}\} = \frac{E\{N\}}{|B|} \overset{\text{def}}{=} \frac{\lambda|B|}{|B|} = \lambda \text{ - forv. rett}$$

$$Var\{\hat{\lambda}\} = \frac{Var\{N\}}{|B|^2}$$

$$= \frac{1}{|B|^2}\sum_i i^2 \, Prob\{N(B)=i\} - \lambda^2$$

sirkel sentrert i x_i - radius r.

$$\hat{K}(r) = \frac{1}{\hat{\lambda}} \cdot \frac{1}{n}\sum_{i=1}^{n}\left[N(b(x_i, r)) - 1\right]$$

- Forv. skjev $\sim \frac{1}{\hat{\lambda}}$
- Rel. høy varians
- rand. probl.

bedre (dvs. mindre varians) estimator:

$$\hat{K}(r) = \frac{1}{\hat{\lambda}^2} \cdot \sum_{i=1}^{n}\sum_{\substack{j=1 \\ i \neq j}}^{n} \frac{I(|x_i - x_j| < r)}{|B_{x_i} \cap B_{x_j}|} \longleftarrow I(A) = \begin{cases} 1 & \text{hvis A sann} \\ 0 & \text{ellers} \end{cases}$$

B_x er B sentrert i x

OPPGAVE 3 MOSAIKKFELT

$L : \{L_x ; x \in \mathcal{L}_D\}$; $L_x \in \{-1, 1\}$

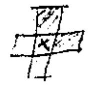
Stasjonært Ising-felt —  N.

a) Markov formulering:

$$\text{Prob}\{L_x = l_x | L_y = l_y ; y \in \mathcal{L}_D ; y \neq x\}$$

$$= \text{konst} \times \exp\{\beta_M \sum_{j \in N_x} l_x l_j\}$$

Naboskap



$$= \text{Prob}\{L_x = l_x | L_y = l_y ; y \in N_x\}$$

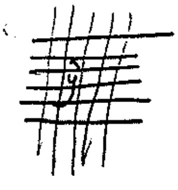
$\beta_M = \beta_G$

Gibbs formulering:

$$\text{Prob}\{L = l\} = \text{konst} \times \exp\{\beta_G \sum_{i \sim j} l_i l_j\}$$

'cliques' og

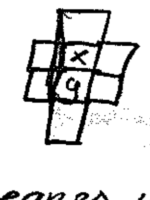
b) Ta utgangspunkt i M-H-algoritmen.



$$q_{ij} = \text{Prob}\{L_x = l_x^j, L_y = l_y^j | L_z = l_z^i ; z \in \mathcal{L}_D^{-xy}\}$$

$$= \frac{\text{Prob}\{L_x = l_x^j, L_y = l_y^j, (L_z = l_z^i ; z \in \mathcal{L}_D^{-xy})\}}{\sum_{x,y} \text{Prob}\{L_x = l_x^j, L_y = l_y^j, (L_z = l_z^i ; z \in \mathcal{L}_D^{-xy}\}}$$

$$= \frac{\exp\{\underbrace{\beta \sum_{x \sim k} l_x l_k + \sum_{\substack{y \sim k \\ k \neq x}} l_y l_k}_{A_{xy}^{ij}} + \sum_{\substack{k \sim l \\ k,l \notin N_{xy}}} l_k^i l_l^i\}}{\sum_{x,y} \exp\{\quad\}}$$



$$= \frac{\exp\{\underbrace{\beta \sum_{x \sim k} l_x l_k + \sum_{\substack{y \sim k \\ k \neq x}} l_y l_k}_{B_{xy}^{ij}}\}}{\sum_{xy} \exp\{\quad\}}$$

kan regnes ut ved $(-1,-1)$, $(-1,1)$, $(1,-1)$ gitt $l_z^i ; z \in \mathcal{L}_D^{-xy}$.

Akseptsannsynligheten er:

$$\alpha_{ij} = \min\left\{1 , \frac{\text{Prob}\{L = l^j\}}{\text{Prob}\{L = l^i\}} \cdot \frac{q_{ji}}{q_{ij}}\right\}$$

MERK! $\{l_z = l_z^j = l_z^i \; ; \; z \in \mathcal{L}_D^{-xy}\}$

$$\alpha_{ij} = \min\left\{1, \frac{\exp\{A_{xy}^{ij}\}}{\sum_{xy} \exp\{A_{xy}^{ij}\}} \cdot \left[\frac{\exp\{A_{xy}^{ji}\}}{\sum_{xy} \exp\{A_{xy}^{ji}\}}\right]^{-1} \frac{q_{ji}}{q_{ij}}\right\}$$

$\{l_z \; ; \; z \in \mathcal{L}_D^{-xy}\}$

$$= \min\left\{1 , q_{ij} \cdot [q_{ji}]^{-1} \cdot \frac{q_{ji}}{q_{ij}}\right\} = \underline{\underline{1}}$$

Algoritme:

Initiale

$L = l^0$

Itererer $i = 1, \ldots$

$l^i = (l_x, l_y, (l_z^{i-1} ; z \in \mathcal{L}_D^{-xy}))$

$\rightsquigarrow \text{Prob}\{L_x = l_x , L_y = l_y \,|\, L_z = l_z^{i-1} ; z \in \mathcal{L}_D^{-xy}\}$

$= \text{Prob}\{L_x = l_x , L_y = l_y \,|\, L_z = l_z^{i-1} ; z \in N_{xy}\}$

se foran