LØSNINGSFORSLAG

EMNE TMA4250 Romlig Statistikk

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Oppgave 1 KONTINUERLIGE FELT

Gauss RF $\{R(x); x \in D \subset \mathbb{R}^2\}$ - stasjonært, isotropt, deriverbart

Parametre: μ, $c(\Delta x)$, $\gamma(\Delta x)$

a)

$$\gamma(\Delta) = \frac{1}{2} Var\{R(x') - R(x'')\}$$

$$= \frac{1}{2}\left[Var\{R(x')\} + Var\{R(x'')\} - 2\, Cov\{R(x'), R(x'')\}\right]$$

$$= c(0) - c(\Delta x)$$

Deriverbarhet i $\{R(x); x \in D\} \Rightarrow \begin{cases} \left.\dfrac{d^2 c(\Delta x)}{d\Delta x^2}\right|_{\Delta x = 0} & \text{eksisterer} \\ \left.\dfrac{d^2 \gamma(\Delta x)}{d\Delta x^2}\right|_{\Delta x = 0} & \text{eksisterer} \end{cases}$

→ Funksjonene er to ganger deriverbare i origo.

b)

$$E\{\hat{\mu}\} = \frac{1}{n}\sum_i^n E\{R^o(x_i)\} = \frac{1}{n}\sum\Big[\underbrace{E\{R(x_i)\}}_{=\mu} + \underbrace{E\{U_i\}}_{=0}\Big] = \underline{\mu}$$

$$E\{\hat{\gamma}(\Delta x)\} = \frac{1}{2n_{\Delta x}} \sum_{ij \in A_{\Delta x}} E\{(R^o(x_i) - R^o(x_j))^2\}$$

$\boxed{\Delta x > 0}$

$$= \frac{1}{2n_{\Delta x}} \sum_{ij \in A_{\Delta x}} Var\{R^o(x_i) - R^o(x_j)\}$$

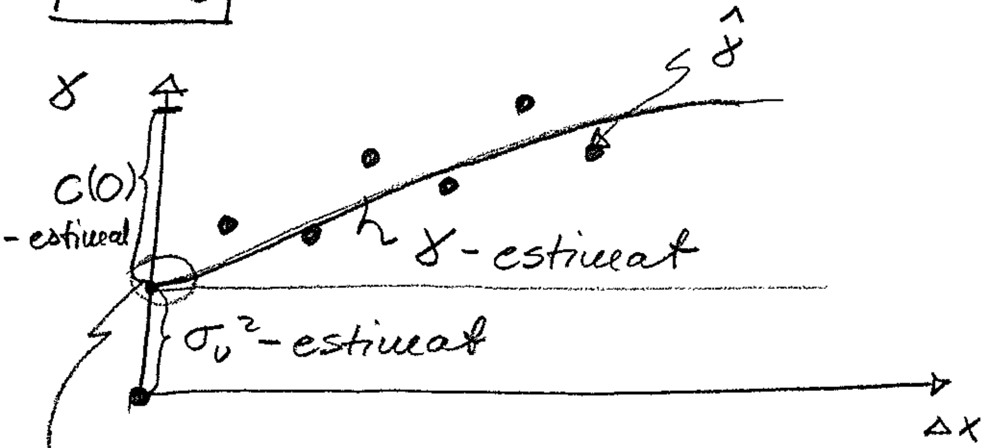
$$= \frac{1}{2n_{\Delta x}} \sum \left[Var\{ R(x_i) - R(x_j) \} + Var\{ U_i - U_j \} \right]$$

$$= \frac{1}{2n_{\Delta x}} \sum \left[\gamma(\Delta x) + \underset{\sigma_U^2}{Var\{U_i\}} + \underset{\sigma_U^2}{Var\{U_j\}} \right]$$

$$= \gamma(\Delta x) + \sigma_U^2$$

$$E\{\hat{\gamma}(\Delta x)\} = \frac{1}{2n_0} \sum_{ij \in A_0} E\{R(x_i) - R(x_j)\} \overset{0}{\equiv} = 0$$

$\boxed{\Delta x = 0}$



må være to ganger deriverbar

$$c(\Delta x)\text{-estimat} = (c(0)\text{-estimat}) - (\gamma(\Delta x)\text{-estimat})$$

c) $\hat{R}(x_0) = \sum_{i=1}^{n} \alpha_i R^{\circ}(x_i)$

Forventningsretthet:

$$E\{\hat{R}(x_0)\} = \mu$$
$$\Updownarrow$$
$$\sum_i \alpha_i E\{R^{\circ}(x_i)\} = \mu$$
$$\Updownarrow$$
$$\sum \alpha_i = 1$$

Prediksjonsvarians:

$$Var\{R(x_0) - \hat{R}(x_0)\} = Var\{R(x_0) - \sum \alpha_i R^{\circ}(x_i)\}$$

$$= Var\{R(x_0)\} - 2\sum \alpha_i Cov\{R(x_0), R^{\circ}(x_i)\}$$

$$+ \sum_i \sum_j \alpha_i \alpha_j Cov\{R^{\circ}(x_i), R^{\circ}(x_j)\}$$

$$= c(0) - 2\sum_i \alpha_i c(\Delta_{0j}) + \left[\sum_i \sum_j \alpha_i \alpha_j c(\Delta_{ij}) + \sum_i \alpha_i^2 \sigma_V^2\right]$$

$\Delta_{ij} = |x_i - x_j|$

Vektene $\alpha = (\alpha_1, .., \alpha_n)$ bestemmes ved:

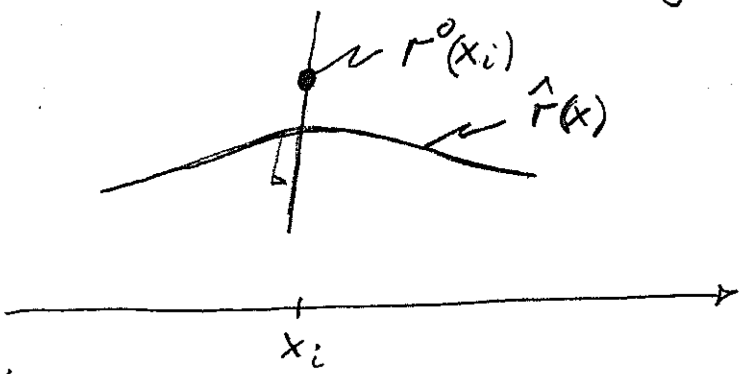
$$\alpha^* = \underset{\substack{\alpha \\ \sum_i \alpha_i = 1}}{\operatorname{argmin}} \left\{ c(0) - 2\sum_i \alpha_i c(\Delta_{0j}) + \sum_i\sum_j \alpha_i \alpha_j c(\Delta_{ij}) + \sum_i \alpha_i^2 \sigma_V^2 \right\}$$

som kan løses med standard Lagrange metode.

Prediktor: $\hat{R}(x_0) = \sum_i \alpha_i^* R^0(x_i)$

Prediksjonsvarians:

$$\sigma_p^2(x_0) = Var\{R(x_0) - \hat{R}(x_0)\} = c(0) - 2\sum \alpha_i^* c(\Delta_{0j}) + \sum_i\sum_j \alpha_i^* \alpha_j^* c(\Delta_{ij}) + \sum_i \alpha_i^{*2} \sigma_V^2\}$$



NB! MERK!

- $\hat{R}(x_i) \neq R^0(x_i)$ — reproduserer ikke observasjon!
- $0 < \sigma_p^2(x_i) < \sigma_V^2$ — prediksjonsvarians ikke 0!

Dette er logisk da det er observasjonsfeil!!!

Oppgave 2 HENDELSES FELT

Punkt RF $\{X_i ; i=1,\dots,N\}$; $D \subset \mathbb{R}^2$

a) Avstand mellom nærmeste-nabopunkt : D

$$\text{Prob}(D > d) = \exp\{-\lambda \pi d^2\}$$

$$F(d) = \text{Prob}(D < d) = 1 - \exp\{-\lambda \pi d^2\}$$

$$f(d) = \frac{dF(d)}{dd} = 2\lambda\pi d \exp\{-\lambda\pi d^2\}$$

b) $d_1, \dots, d_k$ iid $f(d;\lambda)$

$$f(d_1,\dots,d_k;\lambda) = \prod_{i=1}^{k} f(d_i;\lambda)$$

$$= (2\lambda\pi)^k \exp\{-\lambda\pi \sum_i^k d_i^2\} \prod_i^k d_i$$

Loglikelihood:

$$l(\lambda) = k \cdot \ln 2\lambda\pi - \lambda\pi \sum_i^k d_i^2 + \sum_i^k \ln d_i$$

derivér

$$\frac{d\,l(\lambda)}{d\lambda} = \frac{k\, 2\pi}{2\lambda\pi} - \pi \sum_i^k d_i^2$$

Minimér:

$$\frac{k}{\lambda^*} - \pi \sum_i^k d_i^2 = 0$$

$$\Downarrow$$

$$\lambda^* = \left[\frac{\pi}{k} \sum_i^k d_i^2\right]^{-1}$$

MLE-estimator

$$\lambda^* = \left[\frac{\pi}{k} \sum_i^k D_i^2\right]^{-1}$$

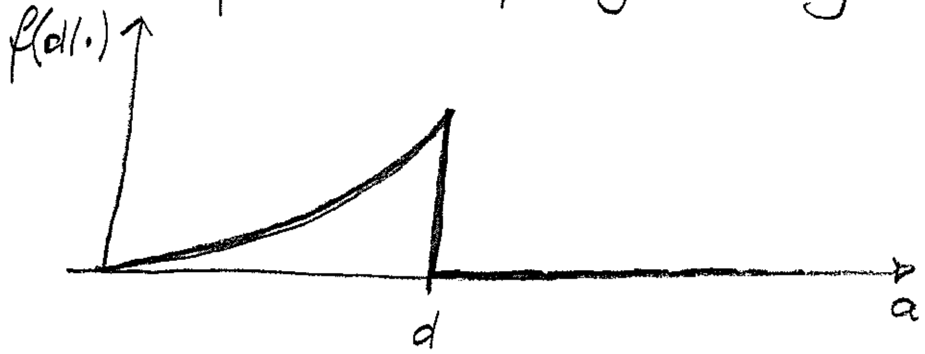
c) Merk at:

$$f(d|d>a) = \begin{cases} 0 & ; \quad d<a \\ \dfrac{f(d)}{1-F(a)} & ; \quad d>a \end{cases}$$

$$= \begin{cases} 0 & ; d<a \\ \dfrac{2\lambda\pi d \exp\{-\lambda\pi d^2\}}{\exp\{-\lambda\pi a^2\}} & ; d>a \end{cases}$$

$$= \begin{cases} 0 & ; d<a \\ 2\lambda\pi \cdot \exp\{\lambda\pi a^2\} d \exp\{-\lambda\pi d^2\} & ; d>a \end{cases}$$

Betrakt $f(d|d>a)$ for gitt d og variabel a:



Observasjoner $d_1, \ldots, d_k$!

Likelihood funksjon:

$$f(d_1, \ldots, d_k; a, \lambda) = \prod_{i=1}^{k} f(d_i | d>a)$$

$$= \begin{cases} 0 & ; a > \min\{d_1, \ldots, d_k\} = d_{min} \\ \left[2\lambda\pi \exp\{\lambda\pi a^2\}\right]^k \exp\{-\lambda\pi \sum_i^k d_i^2\} \prod_{i=1}^{k} d_i & ; \text{else} \end{cases}$$

MERK! Likelihood ikke deriverbar mht a, MEN den er monotont voksende i $[0, d_{min}]$ og 0 for $[d_{min}, \infty]$ dvs.

$$a^* = d_{min} = \min\{d_1, \ldots, d_k\}$$

Log-likelihood:

$$\ell(a^*;\lambda) = k \ln 2\lambda\pi + k\lambda\pi a^{*2} - \lambda\pi \sum_i^k d_i^2 + \sum_i^k \ln d_i$$

Deriverte wrt λ:

$$\frac{d\,\ell(a^*;\lambda)}{d\lambda} = \frac{k2\pi}{2\pi\lambda} + k\pi a^{*2} - \pi \sum_i^k d_i^2$$

Miniḿer:

$$\frac{k}{\lambda^*} + k\pi a^{*2} - \pi \sum_i^k d_i^2 = 0$$

$$\Downarrow$$

$$\lambda^* = \left[\frac{\pi}{k} \sum_i^k d_i^2 - \pi a^{*2}\right]^{-1} = \left[\frac{\pi}{k} \sum_i^k \left[d_i^2 - a^{*2}\right]\right]^{-1}$$

MLE-estimatorer:

$$a^* = \min\{D_1, \dots, D_k\}$$

$$\lambda^* = \left[\frac{\pi}{k} \sum_i^k \left[D_i^2 - a^{*2}\right]\right]^{-1}$$

Oppgave 3 MOSAIKK FELT

Ising RF $L : \{L_x ; x \in \mathcal{L}_D\}$: $L_x \in \{-1, 1\}$

$$\text{Prob}\{L = l\} = \text{const} \times \exp\{\beta \sum_{x \sim y} I(l_x = l_y)\}$$

a) Observert bilde

$o : \{o_x ; x \in \mathcal{L}_D\}$

Stokastisk tolkning:

$O : \{O_x ; x \in \mathcal{L}_D\}$; $O_x \in \{-1, 1\}$

Likelihood modell:

$$\text{Prob}\{O = o \mid L = l\} = \prod_{x \in \mathcal{L}_D} \text{Prob}\{O_x = o_x \mid L_x = l_x\}$$

med

$$\text{Prob}\{O_x = o_x \mid L_x = l_x\} = \begin{cases} p & \text{for } o_x = -l_x \\ (1-p) & \text{for } o_x = l_x \end{cases}$$

Posteriori modell

$$\text{Prob}\{L = l \mid O = o\}$$

$$= \text{const}_1 \times \text{Prob}\{O = o \mid L = l\}\, \text{Prob}\{L = l\}$$

$$= \text{const}_2 \times \prod_{x \in \mathcal{L}_D} \text{Prob}\{O_x = o_x \mid L_x = l_x\} \exp\{\beta \sum_{x \sim y} I(l_x = l_y)\}$$

b) En mulig MCMC-algoritme er 'single-site' oppdatering:

Initier:

l^0 slik at $\text{Prob}\{L = l^0 \mid \theta\} > 0$

Iterer $i = 0, 1, \ldots S$

$$\left.\begin{array}{l} s \rightsquigarrow \text{Uni}[1, \ldots, n_s] \;;\; n_s = \# \mathcal{L}_s \\ l_s^p \rightsquigarrow \text{Uni}[-1, 1] \\ l^p = (l_1^i, \ldots, l_{s-1}^i, l_s^p, l_{s+1}^i, \ldots, l_{n_s}^i) \end{array}\right\} \text{symmetrisk forslagsmatrise}$$

$$\alpha = \min\left\{1, \frac{\text{Prob}\{L = l^p \mid \theta\}}{\text{Prob}\{L = l^i \mid \theta\}}\right\}$$

faller bort her!

set $l^{i+1} = l^p$ med prob α

$l^{i+1} = l^i$ ellers

Da vil

$$\text{Prob}\{L^i = l \mid \theta\} \xrightarrow{i \to \infty} \text{Prob}\{L = l \mid \theta\}$$