SOLUTION / TMA 4250 SPATIAL STATISTICS
Exam May 22. 2006
H. OMRE

PROBLEM 1 Continuous Fields

Consider GRF $\{R(x);\ x \in \mathbb{R}^1\}$:

$$E\{R(x)\} = 0$$

$$Cov\{R(x'), R(x'')\} = \begin{cases} 1 & \text{for } x' = x'' \\ 0 & \text{else} \end{cases}$$

a) Kriging in $R(0)$ given $R(-1) = r_{-1}$ and $R(1) = r_1$.

Kriging is Best Linear Unbiased Predictor under Squared Error Loss;

$$R^*(0) = \alpha_0 + \alpha_1 R(-1) + \alpha_2 R(1)$$

unbiased:

$$E\{R^*(0)\} = \alpha_0 + \alpha_1 E\{R(-1)\} + \alpha_2 E\{R(1)\}$$
$$= \alpha_0 \ [= 0]$$

hence

$$\alpha_0 = 0$$

Prediction variance

$$Var\{R^*(0) - R(0)\}$$
$$= \alpha_1^2 \overset{=1}{Var\{R(-1)\}} + 2\alpha_1\alpha_2 \overset{=0}{Cov\{R(-1), R(1)\}} + \alpha_2^2 \overset{=1}{Var\{R(1)\}}$$
$$- 2\alpha_1 \overset{=0}{Cov\{R(-1), R(0)\}} - 2\alpha_2 \overset{=0}{Cov\{R(1), R(0)\}} + \overset{=1}{Var\{R(0)\}}$$
$$= \alpha_1^2 + \alpha_2^2 + 1$$

Min. variance for $\alpha_1 = \alpha_2 = 0$, ie

$$R^*(0) = 0$$

$$Var\{R^*(0) - R(0)\} = 1$$

which is not surprising since $R(0)$ $R(-1)$ and $R(1)$ are mutually independent.

$$S_d(x) = \int_{x-d/2}^{x+d/2} R(y)\,dy$$

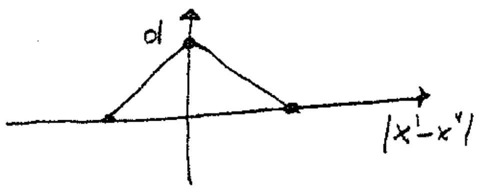
b) $S_d(x)$ is Gaussian since it is a linear combination of $R(x)$ which is Gaussian.

$$E\{S_d(x)\} = \int_{x-d/2}^{x+d/2} E\{R(y)\}\,dx = 0$$

$$\mathrm{Cov}\{S_d(x'), S_d(x'')\} = \int_{x'-d/2}^{x'+d/2}\int_{x''-d/2}^{x''+d/2} \mathrm{Cov}\{R(y), R(z)\}\,dy\,dz$$

$$= \left[\text{overlap of intervals } [x'-d/2, x'+d/2] \text{ and } [x''-d/2, x''+d/2]\right]$$

$$= \begin{cases} d - |x'-x''| & \text{for } |x'-x''| < d \\ 0 & \text{else} \end{cases}$$



c) Kriging:

$$S_2'^{*}(0) = \alpha_0 + \alpha_1 S_1(0) + \alpha_2 S_3(0)$$

$$E\{S_2'^{*}(0)\} = 0 \Rightarrow \alpha_0 = 0$$

$$\mathrm{Var}\{S_2^*(0) - S_2(0)\}$$

$$= \alpha_1^2 \mathrm{Var}\{S_1(0)\} + 2\alpha_1\alpha_2 \mathrm{Cov}\{S_1(0), S_3(0)\} + \alpha_2^2 \mathrm{Var}\{S_3(0)\}$$
$$- 2\alpha_1 \mathrm{Cov}\{S_1(0), S_2(0)\} - 2\alpha_2 \mathrm{Cov}\{S_3(0), S_2(0)\}$$
$$+ \mathrm{Var}\{S_2(0)\}$$

$$\left[\text{NOTE: } \mathrm{Cov}\{S_{d_1}(0), S_{d_2}(0)\} = \int_{-d_1/2}^{d_1/2}\int_{-d_2/2}^{d_2/2} \mathrm{Cov}\{R(y)R(z)\}\,dy\,dz = \min\{d_1, d_2\}\right]$$

$$= \alpha_1^2 + 2\alpha_1\alpha_2 + \alpha_2^2 \cdot 3 - 2\alpha_1 - 2\alpha_2 \cdot 2 + 2$$

$$\frac{d}{d\alpha_\cdot} = 0 \Rightarrow \alpha_1 = \frac{1}{2}\ ;\ \alpha_2 = \frac{1}{2}$$

$$S_2^*(0) = \frac{1}{2} S_1(0) + \frac{1}{2} S_3'(0)$$

$$\mathrm{Var}\{S_2^*(0) - S(0)\} = \underline{\underline{\frac{1}{2}}}$$

PROBLEM 2 Event Fields

Poisson point RF in $\mathbb{R}^2$
Arbitrary location x_0, $R_{(i)}$ - distance to i'th closest point

a) Definition of Poisson point RF:

A, B arbitrary domains in $\mathbb{R}^2$ with $A \cap B = \emptyset$
$N(A)$ - number of points in A
$N(B)$ - number of points in B
Poisson point RF $\Leftrightarrow$ $[N(A), N(B)]$ independent

Poisson point RF on $D \subset \mathbb{R}^2$ with $|D| < \infty$
Poisson point RF $\Leftrightarrow \{X_1, \ldots, X_N; D \subset \mathbb{R}^2\} \Leftrightarrow$
$N = N(D) \sim Poi(\lambda |D|)$ - Poisson pdf par $\lambda |D|$
$[X_1, \ldots, X_N | N]$ - iid $Uni[D]$ - Uniform over D

Given:

$$f(r_{(1)}, \ldots, r_{(k)}) = \begin{cases} [2\lambda\pi]^k \prod_{i=1}^{k} r_{(i)} \exp\{-\lambda\pi r_{(k)}^2\} & ; 0 < r_{(1)} < \cdots < r_{(k)} \\ 0 & \text{else} \end{cases}$$

b) Use induction:
Assum correct for $(k-1)$:

$$f(r_{(1)}, \ldots, r_{(k-1)}) = \begin{cases} [2\lambda\pi]^{k-1} \prod_{i=1}^{k} r_{(i)} \exp\{-\lambda\pi r_{(k-1)}^2\} & ; 0 < r_{(1)} < \cdots < r_{(k-1)} \\ 0 & \text{else} \end{cases}$$

then

$$f(r_{(1)}, \ldots, r_{(k)}) = f(r_{(k)} | r_{(1)}, \ldots, r_{(k-1)}) \cdot f(r_{(1)}, \ldots, r_{(k-1)}) \quad (1)$$

with

$$F(r_{(k)} < r | r_{(1)}, \ldots, r_{(k-1)}) = F(r_{(k)} < r | r_{(k-1)}) = 1 - F(r_{(k)} > r | r_{(k-1)}),$$

$$= 1 - Prob\{N(\mathcal{D}) = 0\} = 1 - \exp\{-\lambda\pi(r_{(k)}^2 - r_{(k-1)}^2)\}$$

$$\Rightarrow f(r_{(k)} | r_{(1)}, \ldots, r_{(k-1)}) = 2\lambda\pi r_{(k)} \exp\{-\lambda\pi(r_{(k)}^2 - r_{(k-1)}^2)\} \quad r > r_{(k-1)}$$

From (1): $f(r_{(1)}, \ldots, r_{(k)}) = [2\lambda\pi]^k \prod_{i=1}^{k} r_{(i)} \exp\{-\lambda\pi r_{(k)}^2\} \quad r_{(k)} > r_{(k-1)}$

Since $f(r_{(1)}) = 2\lambda\pi r_{(1)} \exp\{-\lambda\pi r_{(1)}^2\}$ - is correct,
$f(r_{(1)}, \ldots, r_{(k)})$ is correct by induction

c) Given $r_{(1)}$ and $r_{(3)}$, the likelihood is:

$$f(r_{(1)}, r_{(3)}) = \int_{r_{(1)}}^{r_{(3)}} f(r_{(1)}, r_{(2)}, r_{(3)})\, d r_{(2)}$$

$$= (2\lambda\pi)^3 r_{(1)} r_{(3)} \exp\{-\lambda\pi r_{(3)}^2\} \int_{r_{(1)}}^{r_{(3)}} r\, dr$$

$$= (2\lambda\pi)^3 r_{(1)} r_{(3)} \tfrac{1}{2}(r_{(3)}^2 - r_{(1)}^2) \exp\{-\lambda\pi r_{(3)}^2\}$$

MLE:

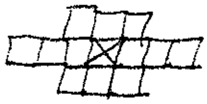
$$\hat{\lambda} = \arg\max_{\lambda} \{ f(r_{(1)}, r_{(3)}; \lambda)\} = \arg\max_{\lambda} \{ \ln f(r_{(1)}, r_{(3)}; \lambda)\}$$

$$\frac{d\,\cdot}{d\lambda} = 0 \Rightarrow 3\frac{1}{\lambda} - \pi r_{(3)}^2 = 0 \Rightarrow$$

$\hat{\lambda} = \dfrac{3}{\pi r_{(3)}^2}$ - which is not unreasonable!

PROBLEM 3 Mosaic Fields

MRF with ∂_s:



a) $$\mathrm{Prob}\{L=\ell\} = \mathrm{const} \times \exp\Big\{-\sum_{c\in G} \xi_c(\ell)\Big\}$$

G - clique system corresponding to ∂

cliques related to ∂_s: