TMA4250 ROMLIG STATISTIKK

LØSNINGSFORSLAG

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PROBLEM 1 CONTINUOUS RANDOM FIELD

$$GRF\{R(x);\, x \in \mathbb{R}^1\}$$

where

$$E\{R(x)\} = \begin{cases} a & \text{for } x < 0 \\ b & \text{for } x \geq 0 \end{cases}$$

$$Cov\{R(x'), R(x'')\} = c(\Delta x) = \exp\{-\Delta x\}$$

with

a, b unknown const

$\Delta x = |x' - x''|$

a) Requirements for pos. def. function $c(\Delta x)$:

$$\sum_i^n \sum_j^n \alpha_i \alpha_j \, c(|x_i - x_j|) \geq 0$$

all configurations $x = (x_1, \ldots, x_n)$

all $\alpha = (\alpha_1, \ldots, \alpha_n)$

all $n > 0$

Observe $R^{o} = (R(x_1), R(x_2), R(x_3), R(x_4))$

with $x^{o} = (x_1, x_2, x_3, x_4) = (-2, -1, 3, 4)$

b) BLUE for a and b

For a:

Consider linear estimator for a:

$$\hat{a} = \sum_{i=1}^{4} \alpha_i R(x_i) \qquad \text{with} \quad \alpha = (\alpha_1, \dots, \alpha_4)$$

Unbiasedness:

$$E\{a - \hat{a}\} = 0$$
$$\Downarrow$$
$$\sum_{i=1}^{4} \alpha_i E\{R(x_i)\} = a$$
$$\Downarrow$$
$$(\alpha_1 + \alpha_2) a + (\alpha_3 + \alpha_4) b = a$$
$$\Downarrow$$
$$(\alpha_1 + \alpha_2) = 1 \qquad (\alpha_3 + \alpha_4) = 0$$

Variance:

$$Var\{a - \hat{a}\} = Var\{\hat{a}\} = \sum_{i=1}^{4} \sum_{j=1}^{4} \alpha_i \alpha_j Cov\{R(x_i), R(x_j)\}$$
$$= \sum_i \sum_j \alpha_i \alpha_j C(\Delta x_{ij})$$

with $\Delta x_{ij} = |x_i - x_j|$

BLUE-weights under quadratic loss are:

$$\alpha^* = \underset{\substack{\alpha \\ \alpha_1 + \alpha_2 = 1 \\ \alpha_3 + \alpha_4 = 0}}{\operatorname{argmin}} \{Var\{a - \hat{a}\}\} = \underset{\alpha}{\operatorname{argmin}} \left\{ \sum_i \sum_j \alpha_i \alpha_j C(\Delta x_{ij}) \right\}$$

Can be solved by Lagrange minimization

BLUE for a with associated estimation variance:

$$\hat{a} = \sum_{i=1}^{4} \alpha_i^* R(x_i)$$

$$Var\{\hat{a}\} = \sum_i \sum_j \alpha_i^* \alpha_j^* C(\Delta x_{ij})$$

For b:

Consider linear estimator for b:

$$\hat{b} = \sum_{i=1}^{4} \beta_i R(x_i) \quad \text{with} \quad \beta = (\beta_1, \beta_2, \beta_3, \beta_4)$$

BLUE-weights under quadratic loss are:

$$\beta^* = \underset{\substack{\beta \\ \beta_1+\beta_2=0 \\ \beta_3+\beta_4=1}}{\operatorname{argmin}} \{ Var\{\beta - \hat{\beta}\} \} = \underset{\beta}{\operatorname{argmin}} \{ \sum_i \sum_j \beta_i \beta_j c(\Delta x_{ij}) \}$$

Can be solved by Lagrange minimization.

BLUE for b with associated estimation variance:

$$\hat{b} = \sum_{i=1}^{4} \beta_i^* R(x_i)$$

$$Var\{\hat{b}\} = \sum_i \sum_j \beta_i^* \beta_j^* c(\Delta x_{ij})$$

Stepheight at $x = 0^-$ is $\Delta = a - b$.

c) Possible estimator $\hat{\Delta} = \hat{a} - \hat{b} = \sum_i \alpha_i^* R(x_i) - \sum_i \beta_i^* R(x_i)$

$$= \sum_i (\alpha_i^* - \beta_i^*) R(x_i)$$

lin. in $R(x)$

$$\hat{\Delta} \sim N(a-b, \sigma_{\hat{\Delta}}^2)$$

$E\{\hat{\Delta}\} = E\{\hat{a}\} - E\{\hat{b}\}$
$\quad\quad\; a \quad - \quad b$

$Var\{\hat{\Delta}\} = \sum_i \sum_j (\alpha_i^* - \beta_i^*)(\alpha_j^* - \beta_j^*) c(\Delta x_{ij})$

0.9 confidensinterval:

$$\left[(\hat{a} - \hat{b}) - \sigma_{\hat{\Delta}} z_{0.05} \,,\, (\hat{a} - \hat{b}) + \sigma_{\hat{\Delta}} z_{0.05} \right]$$

0.05 quantile stand. Gauss

Characteristics of α^*:

$$\mathcal{L}_\alpha = \sum_i \sum_j \alpha_i \alpha_j c_{ij} + \lambda_1^\alpha (\alpha_1 + \alpha_2 - 1) + \lambda_2^\alpha (\alpha_3 + \alpha_4)$$

$$\frac{d\mathcal{L}_\alpha}{d\cdot} = 0 \Rightarrow \left.\begin{aligned} &\sum_j \alpha_j c_{1j} + \lambda_1^\alpha = 0 \\ &\sum_j \alpha_j c_{2j} + \lambda_1^\alpha = 0 \\ &\sum_j \alpha_j c_{3j} + \lambda_2^\alpha = 0 \\ &\sum_j \alpha_j c_{4j} + \lambda_2^\alpha = 0 \\ &\alpha_1 + \alpha_2 = 1 \\ &\alpha_3 + \alpha_4 = 0 \end{aligned}\right\} \quad \begin{aligned} &\sum_j \alpha_j (c_{1j} - c_{2j}) = 0 \\ &\sum_j \alpha_j (c_{3j} - c_{4j}) = 0 \\ &\alpha_2 = 1 - \alpha_1 \\ &\alpha_4 = -\alpha_3 \end{aligned}$$

Characteristics of β^*:

$$\mathcal{L}_\beta = \sum_i \sum_j \beta_i \beta_j c_{ij} + \lambda_1^\beta (\beta_1 + \beta_2) + \lambda_2^\beta (\beta_3 + \beta_4 - 1)$$

$$\frac{d\mathcal{L}_\beta}{d\cdot} = 0 \Rightarrow \left.\begin{aligned} &\sum_j \beta_j c_{1j} + \lambda_1^\beta = 0 \\ &\sum_j \beta_j c_{2j} + \lambda_1^\beta = 0 \\ &\sum_j \beta_j c_{3j} + \lambda_2^\beta = 0 \\ &\sum_j \beta_j c_{4j} + \lambda_2^\beta = 0 \\ &\beta_1 + \beta_2 = 0 \\ &\beta_3 + \beta_4 = 1 \end{aligned}\right\} \quad \begin{aligned} &\sum_j \beta_j (c_{1j} - c_{2j}) = 0 \\ &\sum_j \beta_j (c_{3j} - c_{4j}) = 0 \\ &\beta_2 = -\beta_1 \\ &\beta_4 = 1 - \beta_3 \end{aligned}$$

Krav til BLUE for $\Delta^+ = \sum_i z_i R(x_i)$

$$z^* = \operatorname{argmax}\left\{ \sum_i \sum_j z_i z_j c_{ij} \right\} \qquad \mathrm{Var}\{\Delta^+\} = \sum_i \sum_j z_i z_j c_{ij}$$

$$\left.\begin{array}{l} z_1 + z_2 = 1 \\ z_3 + z_4 = -1 \end{array}\right\} \qquad \begin{array}{l} E\{\Delta - \Delta^+\} = 0 \Rightarrow \\ a - b = (z_1 + z_2)a + (z_3 + z_4)b \end{array}$$

$$L_z = \sum_i \sum_j z_i z_j c_{ij} + \lambda_1^z (z_1 + z_2 - 1) + \lambda_2^z (z_3 + z_4 + 1)$$

$$\frac{dL_z}{d\,\cdot} = 0 \Rightarrow \left.\begin{array}{l} \sum_j z_j c_{1j} + \lambda_1^z = 0 \\ \sum_j z_j c_{2j} + \lambda_1^z = 0 \\ \sum_j z_j c_{3j} + \lambda_2^z = 0 \\ \sum_j z_j c_{4j} + \lambda_2^z = 0 \\ z_1 + z_2 = 1 \\ z_3 + z_4 = -1 \end{array}\right\} \qquad \begin{array}{l} \sum_j z_j (c_{1j} - c_{2j}) = 0 \\ \sum_j z_j (c_{3j} - c_{4j}) = 0 \\ z_2 = 1 - z_1 \\ z_4 = -1 - z_3 \end{array}$$

These expressions uniquely define BLUE for Δ!

Set $z_i = \alpha_i - \beta_i$ and insert:

$$\sum_j (\alpha_j - \beta_j)(c_{1j} - c_{2j}) = \overbrace{\sum_j \alpha_j (c_{1j} - c_{2j})}^{=0} - \overbrace{\sum_j \beta_j (c_{1j} - c_{2j})}^{=0} = 0$$

$$\sum_j (\alpha_j - \beta_j)(c_{3j} - c_{4j}) = \overbrace{\sum_j \alpha_j (c_{3j} - c_{4j})}^{=0} - \overbrace{\sum_j \beta_j (c_{3j} - c_{4j})}^{=0} = 0$$

$$\alpha_2 - \beta_2 = 1 - \alpha_1 + \beta_1 = \alpha_2 - \beta_2$$

$$\alpha_4 - \beta_4 = -1 - \alpha_3 + \beta_3 = \alpha_4 - \beta_4$$

Hence $z_i^* = \alpha_i^* - \beta_i^*$ is the BLUE-weights.

$$\Delta^+ = \hat{\Delta} = \sum_i (\alpha_i^* - \beta_i^*) R(x_i) \ !$$

Conclusion:

The estimator $\hat{\Delta} = \hat{a} - \hat{b}$ is the BLUE for Δ!!

PROBLEM 2 EVENT RANDOM FIELD

Homogeneous PRF intensity λ on $\mathcal{D} \subset \mathbb{R}^2$

Area of $\mathcal{D}$ is ~~$a < \infty$~~ – known constant

No of points in $\mathcal{D}$ is N – RV

No of obs. points in $\mathcal{D}$ with obs. prob p is O – RV

a) Definition of PRF $\{X_1, \ldots, X_N ; \mathcal{D} \subset \mathbb{R}^2\}$

Alt 1:

$$N \rightsquigarrow \text{Poisson}(\lambda a) = \frac{(\lambda a)^n}{n!} e^{-\lambda a} \quad ; \; n = 0, 1, 2, \ldots$$

$$[X_1, \ldots, X_N \mid N = n] \rightsquigarrow \prod_{i=1}^{n} \text{Uni}[\mathcal{D}] = \prod_{i=1}^{n} \frac{1}{a}$$

Alt 2:

For all $\mathcal{B}$ in $\mathcal{D}$: $N \rightsquigarrow \text{Poisson}(\lambda a)$

For all $\mathcal{B}_1, \ldots, \mathcal{B}_\ell$, for all ℓ, disjoint in $\mathcal{D}$:

$(N_{\mathcal{B}_1}, \ldots, N_{\mathcal{B}_\ell})$ independent

where $N_{\mathcal{B}}$ is no. of points in $\mathcal{B}$.

b)

No of points in $\mathcal{D}$ – N – prior pdf

$$p(n) = \frac{(\lambda a)^n}{n!} e^{-\lambda a} \quad \text{– Poisson}$$

No of obs. in $\mathcal{D}$ given $N=n$ – $[O \mid N=n]$ – likelihood

$$p(o \mid n) = \begin{cases} \binom{n}{o} p^o (1-p)^{n-o} & ; \; o \le n \\ 0 & ; \; o > n \end{cases} \quad \text{Binomial}$$

Focus on

No of points in $\mathcal{D}$ given $O = o$ – $[N \mid O = o]$ – post.

$$p(n \mid o) = \frac{p(o \mid n)\, p(n)}{p(o)}$$

Note that

$$p(\sigma) = \sum_{n=0}^{\infty} p(\sigma|n)\,p(n)$$

$$= \sum_{n=\sigma}^{\infty} p(\sigma|n)\,p(n)$$

$$= \frac{1}{\sigma!} p^{\sigma} \cdot e^{-\lambda a} \sum_{n=\sigma}^{\infty} \frac{n!}{(n-\sigma)!} (1-p)^{n-\sigma} \frac{(\lambda a)^n}{n!}$$

$$= \frac{1}{\sigma!} p^{\sigma} e^{-\lambda a} (\lambda a)^{\sigma} \underbrace{\sum_{n=\sigma}^{\infty} \frac{[(1-p)\lambda a]^{n-\sigma}}{(n-\sigma)!}}_{e^{(1-p)\lambda a}}$$

$$= \frac{(p\lambda a)^{\sigma}}{\sigma!} e^{-p\lambda a}$$

Hence:

$$p(n|\sigma) = \begin{cases} \dfrac{n!}{\sigma!(n-\sigma)!} p^{\sigma} (1-p)^{n-\sigma} \dfrac{(\lambda a)^n}{n!} e^{-\lambda a} \dfrac{\sigma!}{(p\lambda a)^{\sigma}} e^{p\lambda a} & ; n \geq \sigma \\ 0 & ; n < \sigma \end{cases}$$

$$= \begin{cases} \dfrac{[(1-p)\lambda a]^{n-\sigma}}{(n-\sigma)!} e^{-(1-p)\lambda a} & ; n \geq \sigma \\ 0 & n < \sigma \end{cases}$$

NOTE that the difference between n and σ, $(n-\sigma)$, follows a Poisson $((1-p)\lambda a)$ which corresponds to the points not observed.
In particular $p=1$, exact observations give:

$$p(n|\sigma) = \begin{cases} 0 & n > \sigma \\ 1 & n = \sigma \\ 0 & n < \sigma \end{cases}$$

PROBLEM 3   MOSAIC RANDOM FIELD

$L : \{L_x ; x \in L_\emptyset\}$ – MRF $\qquad L_\emptyset : (100 \times 100)$

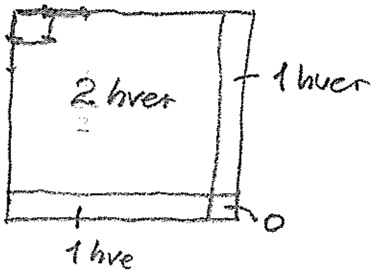
$L_x \in \{-1, 1\}$

$$\mathrm{Prob}\{L = l\} = \mathrm{const} \cdot \exp\{\beta \sum_{\langle u,v \rangle} l_u l_v\}$$

a) Most and least probable outcomes of L.

Note that $\# \langle u,v \rangle =$

$$(99 \times 99) \times \frac{4}{2} = 19602$$
$$(2 \times 99) \times \frac{2}{2} = 198$$
$$(1 \times 1) \times 0 = 0$$
$$\underline{\underline{19800}}$$



Notation ☒ = -1 $\qquad$ □ = 1

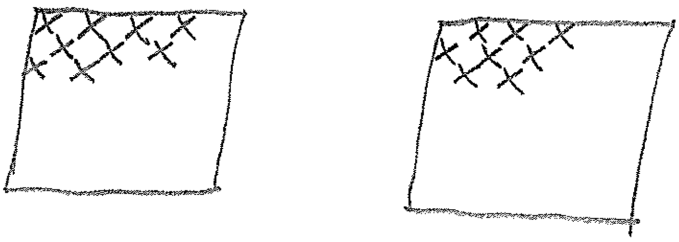
<u>Most probable:</u>

all $l_u l_v = 1$

$$\sum_{\langle u,v \rangle} l_u l_v = 19800$$

$$\mathrm{Prob}\{L = l\} = \mathrm{const} \times \exp\{19800\beta\} = \mathrm{const}\, \underbrace{\exp\{\beta\}}_{>1}{}^{19800}$$

<u>Least probable:</u>



all $l_u l_v = -1$

$$\sum_{\langle u,v \rangle} l_u l_v = -19800$$

$$\mathrm{Prob}\{L = l\} = \mathrm{const} \times \exp\{-19800\beta\} = \mathrm{const}\, \frac{1}{\underbrace{\exp\{\beta\}}_{>1}{}^{19800}}$$

Markov model:

$$\text{Prob}\{L_s = l_s \mid L_{-s} = l_{-s}\} = \frac{\text{Prob}\{L = (l_s, l_{-s})\}}{\sum_{l_s' \in \{-1,1\}} \text{Prob}\{L = (l_s', l_{-s})\}}$$

$$= \frac{\text{const} \cdot \exp\{\beta \sum_{\substack{\langle u,v\rangle \\ u,v \neq s}} l_u l_v\} \exp\{\beta \sum_{\langle s,t\rangle} l_s l_t\}}{\sum_{l_s' \in \{-1,1\}} \text{const} \cdot \exp\{\beta \sum_{\substack{\langle u,v\rangle \\ u,v \neq s}} l_u l_v\} \exp\{\beta \sum_{\langle s,t\rangle} l_s' l_t\}}$$

$$= \frac{\exp\{\beta \sum_{\langle s,t\rangle} l_s l_t\}}{\exp\{-\beta \sum_{\langle s,t\rangle} l_t\} + \exp\{\beta \sum_{\langle s,t\rangle} l_t\}} \quad ; \text{ all } s \in \mathcal{L}_D$$

The major differences between the Gibbs & Markov models are:

- the former is joint pdf
  - the latter uni-conditional pdfs all $s \in \mathcal{L}_D$
- the former has non-computable const
  - the latter has computeable const